A Handbook of Applied EM

By: Oameed Noakoasteen

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Time-Harmonic Electrodynamics
Maxwell's Eq. and their Complete Solutions

Maxwell's Equations

Differential Form

$$\nabla \cdot \boldsymbol{D} = \rho_e$$

$$\nabla \times \boldsymbol{E} = -\boldsymbol{M} - \frac{\partial}{\partial t}\boldsymbol{B}$$

$$\nabla \cdot \boldsymbol{B} = \rho_m$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial}{\partial t}\boldsymbol{D}$$

Integral Form

$$\oiint_S \boldsymbol{D} \cdot d\boldsymbol{S} = Q_e$$

$$\oint_C \boldsymbol{E} \cdot d\boldsymbol{l} = -\iint_S \boldsymbol{M} \cdot d\boldsymbol{S} - \frac{\partial}{\partial t}\iint_S \boldsymbol{B} \cdot d\boldsymbol{S}$$

$$\oiint_S \boldsymbol{B} \cdot d\boldsymbol{S} = Q_m$$

$$\oint_C \boldsymbol{H} \cdot d\boldsymbol{l} = \iint_S \boldsymbol{J} \cdot d\boldsymbol{S} - \frac{\partial}{\partial t}\iint_S \boldsymbol{D} \cdot d\boldsymbol{S}$$

Boundary Conditions

$$\boldsymbol{n}_{12} \cdot (\boldsymbol{D}_2 - \boldsymbol{D}_1) = \rho_{es}$$
$$\boldsymbol{n}_{12} \cdot (\boldsymbol{B}_2 - \boldsymbol{B}_1) = \rho_{ms}$$
$$\boldsymbol{n}_{12} \times (\boldsymbol{E}_2 - \boldsymbol{E}_1) = -\boldsymbol{M}_s$$
$$\boldsymbol{n}_{12} \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = \boldsymbol{J}_s$$

Constitutive Relations

$$\boldsymbol{D} = \epsilon \boldsymbol{E},\ \boldsymbol{B} = \mu \boldsymbol{H},\ \boldsymbol{J} = \sigma \boldsymbol{E}$$

The Time-Harmonic Fields

Under the time-harmonic assumption, the time dependance of any electromagnetic quantity $\psi(\boldsymbol{r}, t)$ is of the form $e^{j\omega t}$, i.e., $\psi(\boldsymbol{r}, t) = \mathrm{Re}\left(\Psi(\boldsymbol{r})\, e^{j\omega t}\right)$. Therefore, the differential form of maxwell's equations can be re-written as:

$$\nabla \cdot \boldsymbol{D} = \rho_e$$
$$\nabla \times \boldsymbol{E} = -\boldsymbol{M} - j\omega \boldsymbol{B}$$
$$\nabla \cdot \boldsymbol{B} = \rho_m$$
$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + j\omega \boldsymbol{D}$$

Complete Time-Harmonic Solutions

$$\boldsymbol{E} = -j\omega \boldsymbol{A} - j\frac{1}{\omega\mu\epsilon}\nabla(\nabla \cdot \boldsymbol{A}) - \frac{1}{\epsilon}\nabla \times \boldsymbol{F}$$

$$\boldsymbol{H} = \frac{1}{\mu}\nabla \times \boldsymbol{A} - j\omega \boldsymbol{F} - j\frac{1}{\omega\mu\epsilon}\nabla(\nabla \cdot \boldsymbol{F})$$

- The Vector Potentials

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$$
$$\nabla^2 \boldsymbol{F} + k^2 \boldsymbol{F} = -\epsilon \boldsymbol{M}$$

The vector potentials $\boldsymbol{A}$ and $\boldsymbol{F}$ are solutions to their respective **wave equations**. The inhomogeneous case describes the **radiation** problem, and the homogeneous case describes the **propagation** problem.

- $k^2 \equiv \omega^2 \mu\epsilon,\ \eta_0 \equiv \sqrt{\frac{\mu_0}{\epsilon_0}},\ c_0 \equiv \frac{1}{\sqrt{\mu_0 \epsilon_0}}$

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Time-Harmonic Electrodynamics
Conservation of Energy and Polarization

Conservation of Energy

- **The Power Balance Equation**

$$P_s = P_e + P_d + j2\omega\left(\overline{W_m} - \overline{W_e}\right)$$

  - Supplied Complex Power (in Watts)

$$P_s \equiv \iiint_V -\frac{1}{2}\left(\boldsymbol{H}^* \cdot \boldsymbol{M}_{\text{imp}} + \boldsymbol{E} \cdot \boldsymbol{J}^*_{\text{imp}}\right) dv$$

  - Exiting Complex Power (in Watts)

$$P_e \equiv \oiint_S \left(\frac{1}{2}\boldsymbol{E} \times \boldsymbol{H}^*\right) \cdot d\boldsymbol{S}$$

  - Dissipated Real Power (in Watts)

$$P_d \equiv \iiint_V \frac{1}{2}\sigma_{eff}\boldsymbol{E} \cdot \boldsymbol{E}^* \, dv$$

  - Time-Averaged Magnetic Energy (in Joules)

$$\overline{W_m} \equiv \iiint_V \frac{1}{4}\mu\boldsymbol{H} \cdot \boldsymbol{H}^* \, dv$$

  - Time-Averaged Electric Energy (in Joules)

$$\overline{W_e} \equiv \iiint_V \frac{1}{4}\epsilon'\boldsymbol{E} \cdot \boldsymbol{E}^* \, dv$$

- **The Average Power Density: The Poynting Vector**

$$\boldsymbol{P}_{\text{av}} \equiv \text{Re}\left(\frac{1}{2}\boldsymbol{E} \times \boldsymbol{H}^*\right)$$

Polarization

- Polarization is the curve traced out (as a function of time) by the endpoint of the instantaneous electric field vector. The field must be observed along the direction of propagation as it travels away from the observer.
- In general, the electric field of a propagating wave can have two components with arbitrary magnitudes and arbitrary phase difference $\Delta\phi$. Therefore, in general, the polarization of a propagating wave is **elliptical**, and the **sense-of-rotation** can be either **right-handed** (RH) or **left-handed** (LH). The **axial ratio** (AR) is defined as the ratio of the major axis of the ellipse to its minor axis. The magnitude of the axial ratio is greater than one, i.e., $|\text{AR}| > 1$, and its sign is negative for a RH wave and is positive for a LH wave.
- When the electric field of a propagating wave has two components with equal magnitudes and $\Delta\phi = 90°$, the ellipse reduces to a circle; Therefore, in this case, the polarization of the propagating wave is **circular** and $|\text{AR}| = 1$.
  For a circularly polarized uniform plane wave that is propagating along the z axis in the positive direction, the right-handed (RHCP) configuration can be expressed as $\boldsymbol{E}_0 = E_0\left(j\boldsymbol{a}_x + \boldsymbol{a}_y\right)$ and the left-handed (RHCP) configuration can be expressed as $\boldsymbol{E}_0 = E_0\left(j\boldsymbol{a}_x - \boldsymbol{a}_y\right)$.
- When the electric field of a propagating wave has only one component or has two components with arbitrary magnitudes and $\Delta\phi = 0°$ or $\Delta\phi = 180°$, the ellipse reduces to a line; Therefore, in this case, the polarization of the propagating wave is **linear** and $|\text{AR}| = \infty$.
  A linearly polarized uniform plane wave that is propagating along the z axis in the positive direction, i.e., $\boldsymbol{E}_0 = E_{0x}\boldsymbol{a}_x + E_{0y}\boldsymbol{a}_y$, can be decomposed into the sum of an RHCP and an LHCP wave, i.e., $\boldsymbol{E}_0 = E_R\left(j\boldsymbol{a}_x + \boldsymbol{a}_y\right) + E_L\left(j\boldsymbol{a}_x - \boldsymbol{a}_y\right)$.

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Time-Harmonic Electrodynamics
Radiation in the Far-Field Region of a Source

Radiation Regions

- The space surrounding a radiating source is divided into three distinct regions, the boundaries of which are defined by two concentric spheres.

  The outer most region, i.e., $r \geq \frac{2D^2}{\lambda}$, is referred to as the **far-field (fraunhofer) region**. This region is defined by the condition that for any observation point, the phase error introduced by approximating $R \equiv \left|\boldsymbol{r} - \boldsymbol{r}'\right|$ (i.e., the distance between the observation point and any point on the source) by the first two terms of its Tylor expansion does not exceed $\frac{\pi}{8}$ radians.

  The middle region, i.e., $0.62\sqrt{\frac{D^3}{\lambda}} \leq r < \frac{2D^2}{\lambda}$, is referred to as the **radiating near-field (fresnel) region**. This region is defined by the condition that for any observation point, the phase error introduced by approximating $R \equiv \left|\boldsymbol{r} - \boldsymbol{r}'\right|$ by the first three terms of its Tylor expansion does not exceed $\frac{\pi}{8}$ radians.

  Finally, the inner most region, i.e., $r < 0.62\sqrt{\frac{D^3}{\lambda}}$, is referred to as the **capacitive near-field region**.

  In the above description of the boundaries of the three regions, $D \equiv 2r'_{\max}$.

Radiation in the Far-Field Region

- **The TEM^r Mode**

  $$E_r = 0$$

  $$E_\theta = -jk\eta \frac{e^{-jkr}}{4\pi r}\left(N_\theta + \frac{1}{\eta}L_\phi\right)$$

  $$E_\phi = -jk\eta \frac{e^{-jkr}}{4\pi r}\left(N_\phi - \frac{1}{\eta}L_\theta\right)$$

  $$\boldsymbol{H} = \frac{1}{\eta}\boldsymbol{a}_r \times \boldsymbol{E}$$

  - The Radiation Integrals

    $$\boldsymbol{N}(\boldsymbol{r}) = \iiint_{V'} \boldsymbol{J}\left(\boldsymbol{r}'\right) e^{jk\boldsymbol{a}_r \cdot \boldsymbol{r}'}\, dv'$$

    $$\boldsymbol{L}(\boldsymbol{r}) = \iiint_{V'} \boldsymbol{M}\left(\boldsymbol{r}'\right) e^{jk\boldsymbol{a}_r \cdot \boldsymbol{r}'}\, dv'$$

  - The Average Power Density

    $$\boldsymbol{P}_{\text{av}} = \left(\frac{1}{2\eta}\boldsymbol{E} \cdot \boldsymbol{E}^*\right)\boldsymbol{a}_r$$

    The spatial distribution of the average power density at a constant radius in the far-field region of the source is referred to as the **radiation pattern** of that source and is given by the **radiation intensity** which is defined as $U(\theta, \phi) \equiv r^2\left(\boldsymbol{P}_{\text{av}} \cdot \boldsymbol{a}_r\right)$.

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Time-Harmonic Electrodynamics
Propagation in Guiding Structures / Free Space

The TM^z Mode

$$E_x = -j\frac{1}{\omega\mu\epsilon}\frac{\partial^2 A_z}{\partial x \partial z} \qquad H_x = \frac{1}{\mu}\frac{\partial A_z}{\partial y}$$

$$E_y = -j\frac{1}{\omega\mu\epsilon}\frac{\partial^2 A_z}{\partial y \partial z} \qquad H_y = -\frac{1}{\mu}\frac{\partial A_z}{\partial x}$$

$$E_z = -j\frac{1}{\omega\mu\epsilon}\left(\frac{\partial^2}{\partial z^2} + k^2\right) A_z \qquad H_z \equiv 0$$

- $\boldsymbol{A} = A_z \boldsymbol{a}_z,\ \nabla^2 A_z + k^2 A_z = 0$

  $\boldsymbol{F} = \boldsymbol{0}$

$$E_\rho = -j\frac{1}{\omega\mu\epsilon}\frac{\partial^2 A_z}{\partial \rho \partial z} \qquad H_\rho = \frac{1}{\mu}\frac{1}{\rho}\frac{\partial A_z}{\partial \phi}$$

$$E_\phi = -j\frac{1}{\omega\mu\epsilon}\frac{1}{\rho}\frac{\partial^2 A_z}{\partial \phi \partial z} \qquad H_\phi = -\frac{1}{\mu}\frac{\partial A_z}{\partial \rho}$$

$$E_z = -j\frac{1}{\omega\mu\epsilon}\left(\frac{\partial^2}{\partial z^2} + k^2\right) A_z \qquad H_z \equiv 0$$

- $\boldsymbol{A} = A_z \boldsymbol{a}_z,\ \nabla^2 A_z + k^2 A_z = 0$

  $\boldsymbol{F} = \boldsymbol{0}$

The TE^z Mode

$$E_x = -\frac{1}{\epsilon}\frac{\partial F_z}{\partial y} \qquad H_x = -j\frac{1}{\omega\mu\epsilon}\frac{\partial^2 F_z}{\partial x \partial z}$$

$$E_y = \frac{1}{\epsilon}\frac{\partial F_z}{\partial x} \qquad H_y = -j\frac{1}{\omega\mu\epsilon}\frac{\partial^2 F_z}{\partial y \partial z}$$

$$E_z \equiv 0 \qquad H_z = -j\frac{1}{\omega\mu\epsilon}\left(\frac{\partial^2}{\partial z^2} + k^2\right) F_z$$

- $\boldsymbol{A} = \boldsymbol{0}$

  $\boldsymbol{F} = F_z \boldsymbol{a}_z,\ \nabla^2 F_z + k^2 F_z = 0$

$$E_\rho = -\frac{1}{\epsilon}\frac{1}{\rho}\frac{\partial F_z}{\partial \phi} \qquad H_\rho = -j\frac{1}{\omega\mu\epsilon}\frac{\partial^2 F_z}{\partial \rho \partial z}$$

$$E_\phi = \frac{1}{\epsilon}\frac{\partial F_z}{\partial \rho} \qquad H_\phi = -j\frac{1}{\omega\mu\epsilon}\frac{1}{\rho}\frac{\partial^2 F_z}{\partial \phi \partial z}$$

$$E_z \equiv 0 \qquad H_z = -j\frac{1}{\omega\mu\epsilon}\left(\frac{\partial^2}{\partial z^2} + k^2\right) F_z$$

- $\boldsymbol{A} = \boldsymbol{0}$

  $\boldsymbol{F} = F_z \boldsymbol{a}_z,\ \nabla^2 F_z + k^2 F_z = 0$

Plane Wave: The TEM^z Mode

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{-jkz}$$

$$\boldsymbol{H} = \frac{\omega\epsilon}{k}\boldsymbol{a}_z \times \boldsymbol{E}$$

- $\boldsymbol{E}_0 = -\boldsymbol{\nabla}\phi_e,\ \nabla^2\phi_e(x,y) = 0$ or $\nabla^2\phi_e(\rho,\phi) = 0$

  The **static potential** ϕ_e is the solution to the **laplace equation** on the plane that is transverse to the z axis, i.e., the $x-y$ plane in the rectangular coordinate system or the $\rho-\phi$ plane in the cylindrical coordinate system.

Plane Wave: The Uniform Plane Wave

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{-j\boldsymbol{k}\cdot\boldsymbol{r}}$$

$$\boldsymbol{H} = \frac{1}{\omega\mu}\boldsymbol{k} \times \boldsymbol{E}$$

- The vector $\boldsymbol{k} = k_x\boldsymbol{a}_x + k_y\boldsymbol{a}_y + k_z\boldsymbol{a}_z$ represents the (arbitrary) direction of propagation. The components of $\boldsymbol{k}$ must satisfy $k_x^2 + k_y^2 + k_z^2 = k^2$. The vector $\boldsymbol{E}_0 = E_{0x}\boldsymbol{a}_x + E_{0y}\boldsymbol{a}_y + E_{0z}\boldsymbol{a}_z$ represents the (arbitrary) constant electric field on the plane that is transverse to $\boldsymbol{k}$. The vectors $\boldsymbol{k}$ and $\boldsymbol{E}_0$ must satisfy $\boldsymbol{k}\cdot\boldsymbol{E}_0 = 0$.

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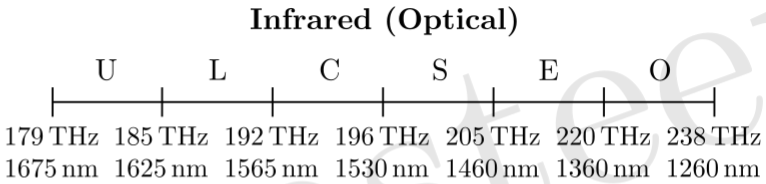
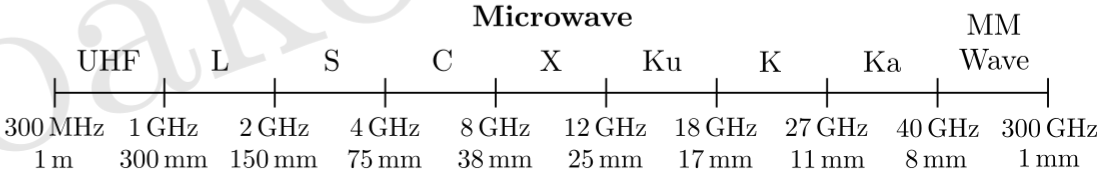
Time-Harmonic Electrodynamics
Material Media and Frequency Bands

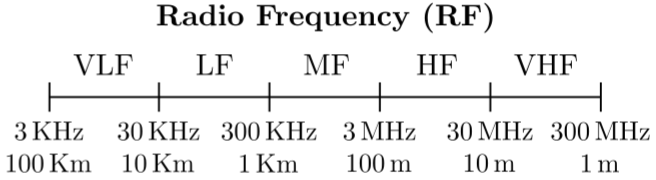
Material Media

- In **dielectric** (insulator) media, the electric flux density is the sum of the electric field and the **electric polarization vector**, i.e., $\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P}$ in which ϵ_0 is the permittivity of free space.
  In a **non-polar** medium, in the absence of an applied electric field, no individual electric dipoles exist and, therefore, the electric polarization vector is zero, i.e., $\boldsymbol{P} = \boldsymbol{0}$. However, when an electric field is applied, $\boldsymbol{P} \neq \boldsymbol{0}$.
  In a **polar** medium, in the absence of an applied electric field, individual electric dipoles do exist, but due to their random orientation, the electric polarization vector is zero, i.e., $\boldsymbol{P} = \boldsymbol{0}$. However, when an electric field is applied, $\boldsymbol{P} \neq \boldsymbol{0}$.
  In an **electret** medium, in the absence of an applied electric field, the electric polarization vector is non-zero, i.e., $\boldsymbol{P} \neq \boldsymbol{0}$.
  When the electric polarization vector is non-zero due to an applied filed, it is given by $\boldsymbol{P} = \epsilon_0 \chi_e \boldsymbol{E}$ in which χ_e is the **electric susceptibility**.
- In **magnetic** media, the magnetic flux density is the sum of the magnetic field and the **magnetic polarization vector**, i.e., $\boldsymbol{B} = \mu_0 \boldsymbol{H} + \mu_0 \boldsymbol{M}$ in which μ_0 is the permeability of free space.
  In **ferromagnetic** and **ferrimagnetic** (ferrite) media, in the absence of an applied magnetc field, individual magnetic dipoles do exist, but due to their random orientation, the magnetic polarization vector is zero, i.e., $\boldsymbol{M} = \boldsymbol{0}$; When a magnetic field is applied, $\boldsymbol{M} \neq \boldsymbol{0}$ and is given by $\boldsymbol{M} = \chi_m \boldsymbol{H}$ in which χ_m is the **magnetic susceptibility**. However, when the applied magnetic field is removed, the individual magnetic dipoles do not attain a random orientation and, as a result, the magnetic polarization vector remains non-zero.
- A medium is **linear** if its constitutive parameters are not a function of the strength of the applied field (otherwise it's non-linear); It's **isotropic** if its constitutive parameters are not a function of the direction of the applied field (otherwise it's anisotropic). It's **homogeneous** if its constitutive parameters are not a function of position (otherwise it's non-homogeneous). It's **non-dispersive** if its constitutive parameters are not a function of frequency (otherwise it's dispersive).

Frequency Bands


Infrared (Optical)
U L C S E O
179 THz 185 THz 192 THz 196 THz 205 THz 220 THz 238 THz
1675 nm 1625 nm 1565 nm 1530 nm 1460 nm 1360 nm 1260 nm


Microwave
UHF L S C X Ku K Ka MM Wave
300 MHz 1 GHz 2 GHz 4 GHz 8 GHz 12 GHz 18 GHz 27 GHz 40 GHz 300 GHz
1 m 300 mm 150 mm 75 mm 38 mm 25 mm 17 mm 11 mm 8 mm 1 mm



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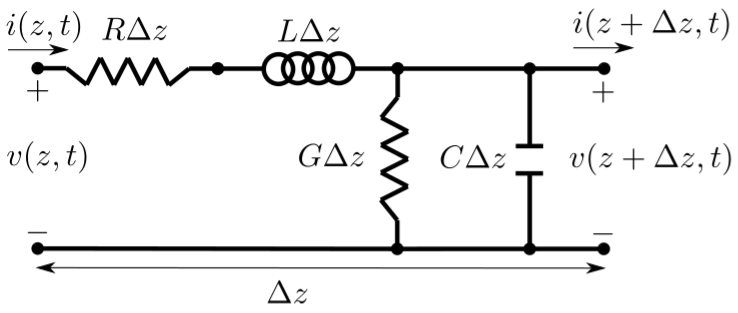
The Transmission Line Model
Telegrapher's Eq. and their General Solutions

Telegrapher's Equations

■ 

$$-\frac{\partial v(z,t)}{\partial z} = L\frac{\partial i(z,t)}{\partial t} + Ri(z,t)$$

$$-\frac{\partial i(z,t)}{\partial z} = C\frac{\partial v(z,t)}{\partial t} + Gv(z,t)$$

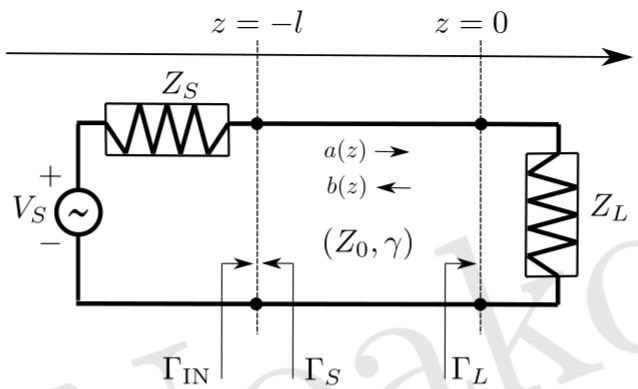
■ **The Time-Harmonic Waves**

Under the time-harmonic assumption, the time dependance of the (voltage or current) waves $\psi(z,t)$ is of the form $e^{j\omega t}$, i.e., $\psi(z,t) = \mathrm{Re}\left(\Psi(z)e^{j\omega t}\right)$. Therefore, the telegrapher's equations can be re-written as:

$$-\frac{\partial V(z)}{\partial z} = (R+j\omega L)\, I(z)$$

$$-\frac{\partial I(z)}{\partial z} = (G+j\omega C)\, V(z)$$

General Time-Harmonic Solutions

■ 

$$V(z) = \frac{1}{\sqrt{R_0}}\left(Z_0^* a(z) + Z_0 b(z)\right)$$

$$I(z) = \frac{1}{\sqrt{R_0}}\left(a(z) - b(z)\right)$$

$$P_{\mathrm{av}}(z) \equiv \mathrm{Re}\left(\frac{1}{2}V(z)\, I(z)^*\right) = \frac{1}{2}\left(|a(z)|^2 - |b(z)|^2\right)$$

▪ Power Waves

$$a(z) = \frac{V^+}{\sqrt{R_0}}e^{-\gamma z}$$

$$b(z) = \frac{V^+}{\sqrt{R_0}}e^{-\gamma z}\left[\frac{R_0}{Z_0}\left(j\frac{X_0}{R_0} + \Gamma_L e^{2\gamma z}\right)\right]$$

■ **Line Parameters**

$$Z_0 = R_0 + jX_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}}$$

$$\gamma = \alpha + j\beta = \sqrt{(R+j\omega L)(G+j\omega C)}$$

$$\Gamma_L \equiv \frac{V^-}{V^+} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$Z_{\mathrm{IN}} \equiv \frac{V(z=-l)}{I(z=-l)} = Z_0\frac{Z_L + Z_0\tanh\gamma l}{Z_0 + Z_L\tanh\gamma l}$$

$$\Gamma_{\mathrm{IN}} \equiv \frac{Z_{\mathrm{IN}} - Z_0}{Z_{\mathrm{IN}} + Z_0} = \Gamma_L e^{-2\gamma l}$$

$$\Gamma_S \equiv \frac{Z_S - Z_0}{Z_S + Z_0}$$

$$\frac{V^+}{V_S} = \frac{1}{2}\left(\frac{1-\Gamma_S}{1-\Gamma_S\Gamma_{\mathrm{IN}}}\right)e^{-\gamma l}$$

$$\mathrm{VSWR} \equiv \frac{\max(|V(z)|)}{\min(|V(z)|)}$$

■ When $\frac{R}{\omega L} \ll 1$ and $\frac{G}{\omega C} \ll 1$, the transmission line is said to be **low-loss**. In this case, $Z_0 = \sqrt{\frac{L}{C}}$ and $\gamma = \frac{1}{2}\left(\frac{R}{Z_0} + GZ_0\right) + j\omega\sqrt{LC}$.
When $R=0$ and $G=0$, the transmission line is said to be **lossless**. In this case, $Z_0 = \sqrt{\frac{L}{C}}$ and $\gamma = j\omega\sqrt{LC}$, $a(z) = V_n^+ e^{-j\beta z}$ and $b(z) = V_n^+\Gamma_L e^{j\beta z}$ in which $V_n^+ \equiv \frac{V^+}{\sqrt{Z_0}}$.

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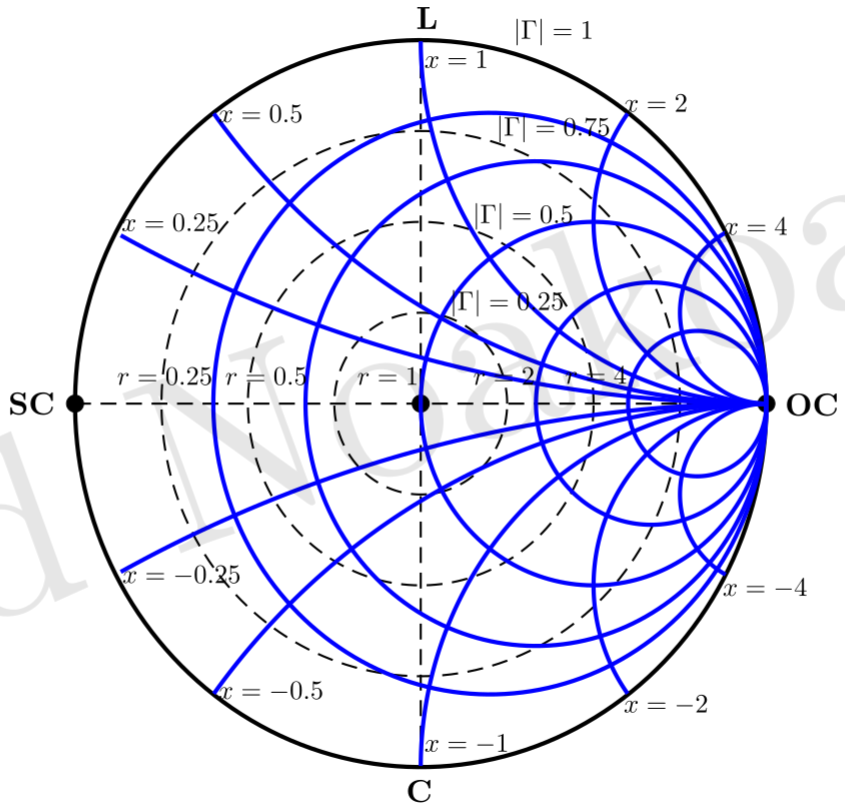
The Transmission Line Model
The Smith Chart

The Impedance Smith Chart

The impedance chart comprises loci of constant normalized **resistance** and constant normalized **reactance** overlaid on the $|\Gamma| \leq 1$ region of the Γ-plane. The normalization is done with respect to the reference impedance Z_0.

The locus of Γ values for which $r \equiv \mathrm{Re}\left(\frac{Z}{Z_0}\right)$ is constant, is a circle with its center (c) and radius (r) given by $c = \frac{r}{r+1}$ and $r = \left|\frac{1}{r+1}\right|$ respectively.

The locus of Γ values for which $x \equiv \mathrm{Im}\left(\frac{Z}{Z_0}\right)$ is constant, is a circle with its center (c) and radius (r) given by $c = 1 + j\frac{1}{x}$ and $r = \left|\frac{1}{x}\right|$ respectively.



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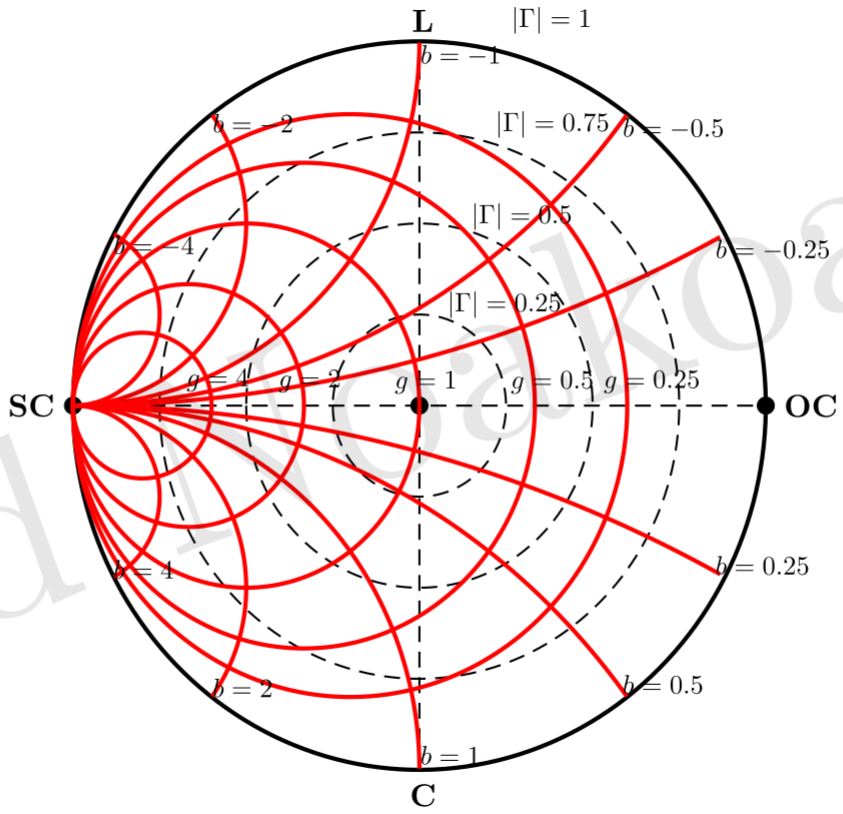
The Transmission Line Model
The Smith Chart

The Admittance Smith Chart

The admittance chart comprises loci of constant normalized **conductance** and constant normalized **susceptance** overlaid on the $|\Gamma| \leq 1$ region of the Γ-plane. The normalization is done with respect to the reference admittance Y_0.

The locus of Γ values for which $g \equiv \text{Re}\left(\frac{Y}{Y_0}\right)$ is constant, is a circle with its center (c) and radius (r) given by $c = -\frac{g}{g+1}$ and $r = \left|\frac{1}{g+1}\right|$ respectively.

The locus of Γ values for which $b \equiv \text{Im}\left(\frac{Y}{Y_0}\right)$ is constant, is a circle with its center (c) and radius (r) given by $c = -1 - j\frac{1}{b}$ and $r = \left|\frac{1}{b}\right|$ respectively.



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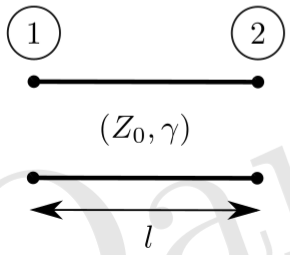
Network Analysis
Scattering Matrix and Power Relations

Scattering Matrix

- In an N-port network, the input and output power waves at the i^{th} port are denoted by $V_{n,i}^{+}$ and $V_{n,i}^{-}$ respectively; The output power wave at the i^{th} port is assumed to be a weighted sum of the input power waves at all ports, i.e., $V_{n,i}^{-} = \sum_{j=1}^{N} S_{ij} V_{n,j}^{+}$. The complex weighing factors S_{ij} are referred to as the **scattering parameters** and are determined through measurement.
  Since $S_{ik} = \frac{V_{n,i}^{-}}{V_{n,k}^{+}}$ when $V_{n,j}^{+} = 0$ for all $j \neq k$, it follows that S_{ik} can be measured by terminating all ports $j \neq k$ with their characteristic impedances and exciting the k^{th} port with $V_{n,k}^{+}$.
  The vector containing all N output power waves can be obtained by $\boldsymbol{V}_n^{-} = \boldsymbol{S}\boldsymbol{V}_n^{+}$ in which $\boldsymbol{V}_n^{+}$ is the vector containing all N input power waves and $\boldsymbol{S}$ is the scattering matrix.

- **An Illustrative Example:**
  The Scattering Matrix of a Low-loss Transmission Line



$$\boldsymbol{S} = e^{-\gamma l}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Power and Loss

- The power exiting any port (in dBm) is obtained by subtracting a loss quantity (in dB) from the available power at port one (in dBm). The loss between any two ports is specific to that pair.
  The loss between port one and port one is referred to as the **return loss** (RL). The loss between port two and port one is referred to as the **insertion loss** (IL). The loss between port three and port one is referred to as the **coupling** (C). The loss between port four and port one is referred to as the **isolation** (I).

  - Return Loss

$$\mathrm{RL}^{\mathrm{dB}} \equiv 10\log\left(\frac{P_1^{\text{available}}}{P_1^{\text{exiting}}}\right) = 10\log\left(\frac{1}{|\Gamma_{\mathrm{IN}}|^2}\right)$$

  - Insertion Loss

$$\mathrm{IL}^{\mathrm{dB}} \equiv 10\log\left(\frac{P_1^{\text{available}}}{P_2^{\text{exiting}}}\right)$$

  - As an illustrative example, for a lossless transmission line of length l which is terminated in Z_L, $\mathrm{RL}^{\mathrm{dB}} = 10\log\left(\frac{1}{|\Gamma_L|^2}\right)$ and $\mathrm{IL}^{\mathrm{dB}} = 10\log\left(\frac{1}{1-|\Gamma_L|^2}\right)$.

  - Coupling

$$\mathrm{C}^{\mathrm{dB}} \equiv 10\log\left(\frac{P_1^{\text{available}}}{P_3^{\text{exiting}}}\right)$$

  - Isolation

$$\mathrm{I}^{\mathrm{dB}} \equiv 10\log\left(\frac{P_1^{\text{available}}}{P_4^{\text{exiting}}}\right)$$

  - Directivity

$$\mathrm{D}^{\mathrm{dB}} \equiv 10\log\left(\frac{P_3^{\text{exiting}}}{P_4^{\text{exiting}}}\right) = \mathrm{I}^{\mathrm{dB}} - \mathrm{C}^{\mathrm{dB}}$$

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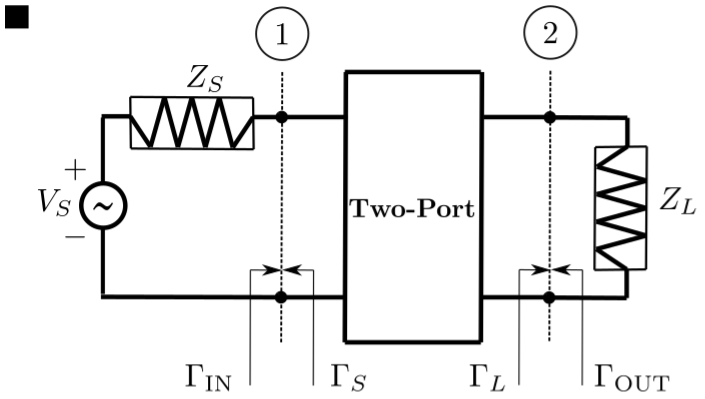
Network Analysis
Two-Port Networks

Two-Ports: The Power Gain



- Input Reflection Coefficient

$$\Gamma_{\text{IN}} \equiv \frac{V_{n,1}^-}{V_{n,1}^+} = \frac{S_{11} - \Delta\Gamma_L}{1 - S_{22}\Gamma_L},\ \Delta \equiv S_{11}S_{22} - S_{12}S_{21}$$

- Output Reflection Coefficient

$$\Gamma_{\text{OUT}} \equiv \frac{V_{n,2}^-}{V_{n,2}^+} = \frac{S_{22} - \Delta\Gamma_S}{1 - S_{11}\Gamma_S},\ \Delta \equiv S_{11}S_{22} - S_{12}S_{21}$$

- Source Mismatch Factor

$$M_S \equiv \frac{P_1^{\text{entering}}}{P_1^{\text{available}}} = \frac{\left(1 - |\Gamma_S|^2\right)\left(1 - |\Gamma_{\text{IN}}|^2\right)}{|1 - \Gamma_S\Gamma_{\text{IN}}|^2}$$

- Load Mismatch Factor

$$M_L \equiv \frac{P_2^{\text{exiting}}}{P_2^{\text{available}}} = \frac{\left(1 - |\Gamma_L|^2\right)\left(1 - |\Gamma_{\text{OUT}}|^2\right)}{|1 - \Gamma_L\Gamma_{\text{OUT}}|^2}$$

- **The Operating Power Gain**

$$G_P \equiv \frac{P_2^{\text{exiting}}}{P_1^{\text{entering}}} = \frac{1}{1 - |\Gamma_{\text{IN}}|^2}\,|S_{21}|^2\,\frac{1 - |\Gamma_L|^2}{|1 - S_{22}\Gamma_L|^2}$$

- **The Available Power Gain**

$$G_A \equiv \frac{P_2^{\text{available}}}{P_1^{\text{available}}} = \frac{M_S}{M_L}\frac{P_2^{\text{exiting}}}{P_1^{\text{entering}}} = \frac{1}{1 - |\Gamma_{\text{OUT}}|^2}\,|S_{21}|^2\,\frac{1 - |\Gamma_S|^2}{|1 - S_{11}\Gamma_S|^2}$$

- **The Transducer Power Gain**

$$G_T \equiv \frac{P_2^{\text{exiting}}}{P_1^{\text{available}}} = M_S\frac{P_2^{\text{exiting}}}{P_1^{\text{entering}}} = \frac{1 - |\Gamma_S|^2}{|1 - \Gamma_{\text{IN}}\Gamma_S|^2}\,|S_{21}|^2\,\frac{1 - |\Gamma_L|^2}{|1 - S_{22}\Gamma_L|^2} = \frac{1 - |\Gamma_S|^2}{|1 - S_{11}\Gamma_S|^2}\,|S_{21}|^2\,\frac{1 - |\Gamma_L|^2}{|1 - \Gamma_{\text{OUT}}\Gamma_L|^2}$$

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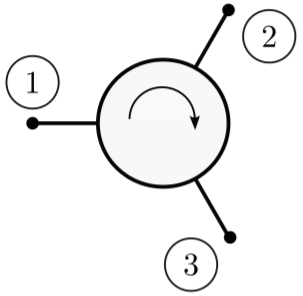
Network Analysis
Three-Port and Four-Port Networks

Three-Ports: The Circulator

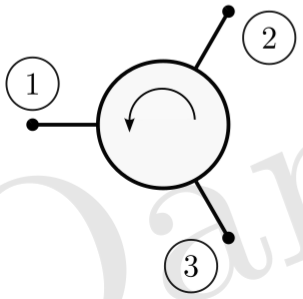
- A circulator is a **non-reciprocal**, **matched** and **lossless** three-port network.

- **The Clockwise Circulator**



$$\boldsymbol{S} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

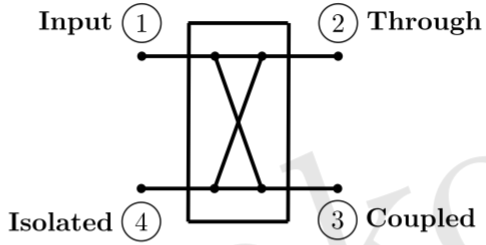
- **The Counterclockwise Circulator**



$$\boldsymbol{S} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Four-Ports: The Directional Coupler

- A directional coupler is a **reciprocal**, **matched** and **lossless** four-port network in which $|S_{41}| = |S_{32}| = 0$.



- **The Antisymmetric Coupler**

$$\boldsymbol{S} = \begin{bmatrix} 0 & \alpha & \beta & 0 \\ \alpha & 0 & 0 & -\beta \\ \beta & 0 & 0 & \alpha \\ 0 & -\beta & \alpha & 0 \end{bmatrix}, \quad \alpha^2 + \beta^2 = 1$$

- The 180° Hybrid

$$\boldsymbol{S} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

- **The Symmetric Coupler**

$$\boldsymbol{S} = \begin{bmatrix} 0 & \alpha & j\beta & 0 \\ \alpha & 0 & 0 & j\beta \\ j\beta & 0 & 0 & \alpha \\ 0 & j\beta & \alpha & 0 \end{bmatrix}, \quad \alpha^2 + \beta^2 = 1$$

$$\boldsymbol{S} = \begin{bmatrix} 0 & \alpha & -j\beta & 0 \\ \alpha & 0 & 0 & -j\beta \\ -j\beta & 0 & 0 & \alpha \\ 0 & -j\beta & \alpha & 0 \end{bmatrix}, \quad \alpha^2 + \beta^2 = 1$$

- The 90° Hybrid

$$\boldsymbol{S} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & j & 0 \\ 1 & 0 & 0 & j \\ j & 0 & 0 & 1 \\ 0 & j & 1 & 0 \end{bmatrix}$$

$$\boldsymbol{S} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & -j & 0 \\ 1 & 0 & 0 & -j \\ -j & 0 & 0 & 1 \\ 0 & -j & 1 & 0 \end{bmatrix}$$

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Nonlinearity, Noise and Dynamic Range

Weakly Nonlinear Circuits

- In a **linear** circuit, the relation between the output and the input is described by a linear function; This means that, if y_1 is the circuit's response to x_1 and y_2 is its response to x_2, then its response to $ax_1 + bx_2$ will be $ay_1 + by_2$. The response of a linear, time-invariant (LTI) circuit includes only those frequencies that were present in the excitation waveform. In other words, LTI circuits don't generate new frequencies.
  In a **weakly nonlinear** circuit, the relation between the output and the input is described by a nonlinear function which can be approximated, using the **Tylor series expansion**, as a linear function around a **bias point**.
  In a **strongly nonlinear** circuit, the relation between the output and the input can only be analyzed by the **harmonic-balance** method or **time-domain** methods.

- **Two-Tone Analysis**

$$
\begin{aligned}
v_i &= V_1 \cos\omega_1 t + V_2 \cos\omega_2 t \\
v_o &= f(v_i) \\
&= \sum_{n=0}^{\infty} a_n v_i^n \\
&\approx a_0 + a_1 v_i + a_2 v_i^2 + a_3 v_i^3 \\
&= a_0 + a_2\left(\frac{1}{2}V_1^2 + \frac{1}{2}V_2^2\right) \\
&+ \left[a_1 V_1 + a_3\left(\frac{3}{4}V_1^3 + \frac{3}{2}V_1 V_2^2\right)\right]\cos\omega_1 t + \left[a_2\left(\frac{1}{2}V_1^2\right)\right]\cos 2\omega_1 t + \left[a_3\left(\frac{1}{4}V_1^3\right)\right]\cos 3\omega_1 t \\
&+ \left[a_1 V_2 + a_3\left(\frac{3}{4}V_2^3 + \frac{3}{2}V_1^2 V_2\right)\right]\cos\omega_2 t + \left[a_2\left(\frac{1}{2}V_2^2\right)\right]\cos 2\omega_2 t + \left[a_3\left(\frac{1}{4}V_2^3\right)\right]\cos 3\omega_2 t \\
&+ [a_2 V_1 V_2]\cos(\omega_1 - \omega_2)\,t + [a_2 V_1 V_2]\cos(\omega_1 + \omega_2)\,t \\
&+ \left[a_3\left(\frac{3}{4}V_1^2 V_2\right)\right]\cos(2\omega_1 - \omega_2)\,t + \left[a_3\left(\frac{3}{4}V_1^2 V_2\right)\right]\cos(2\omega_1 + \omega_2)\,t \\
&+ \left[a_3\left(\frac{3}{4}V_1 V_2^2\right)\right]\cos(\omega_1 - 2\omega_2)\,t + \left[a_3\left(\frac{3}{4}V_1 V_2^2\right)\right]\cos(\omega_1 + 2\omega_2)\,t
\end{aligned}
$$

  - Power Characteristics: P_{ω_1} and $P_{2\omega_1-\omega_2}$

    The output power of the fundamental frequency ω_1 is given by $P_{\omega_1}^{\text{nonlinear}} = a_1^2 P_{\text{in}} + 9a_1 a_3 P_{\text{in}}^2 + \frac{81}{4}a_3^2 P_{\text{in}}^3$. Its linear approximation, denoted by P_{ω_1}, is obtained by keeping only the first term, i.e., $P_{\omega_1} = a_1^2 P_{\text{in}}$.
    The output power of the third-order intermodulation frequency $2\omega_1 - \omega_2$ is given by $P_{2\omega_1-\omega_2} = \frac{9}{4}a_3^2 P_{\text{in}}^3$. Furthermore, $P_{2\omega_1-\omega_2} = \frac{9}{4}\left(\frac{a_3}{a_1^3}\right)^2 P_{\omega_1}^3$.

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Nonlinearity, Noise and Dynamic Range

Weakly Nonlinear Circuits (Cont'd)

Gain Compression

The **1^{dB} compression point** (P1dB) is the power level at which the output power of the fundamental frequency ω_1, i.e., $P_{\omega_1}^{\text{nonlinear}}$, has decreased by 1^{dB} from P_{ω_1}.
The 1^{dB} compression point can be referenced either at the input (in which case it is denoted by IP1dB) or at the output (in which case it is denoted by OP1dB); For each particular application, the maximum is chosen, i.e., $\text{P1dB} = \max(\text{IP1dB}, \text{OP1dB})$.

- $\text{OP1dB}^{\text{dBm}} = \text{IP1dB}^{\text{dBm}} + G^{\text{dB}} - 1^{\text{dB}}$

Third-Order Intermodulation Distortion

The **third-order intercept point** (IP3) is the hypothetical power level at which $P_{2\omega_1-\omega_2} = P_{\omega_1}$.
The third-order intercept point can be referenced either at the input (in which case it is denoted by IIP3) or at the output (in which case it is denoted by OIP3); For each particular application, the maximum is chosen, i.e., $\text{IP3} = \max(\text{IIP3}, \text{OIP3})$.

- $\text{OIP3}^{\text{dBm}} = \text{IIP3}^{\text{dBm}} + G^{\text{dB}}$
- $P_{2\omega_1-\omega_2} = \dfrac{1}{\text{OIP3}^2} P_{\omega_1}^3$
- A cascade connection of components usually has the effect of degrading (lowering) the IP3. For the case of two-stage cascade:

$$\text{OIP3} = \left[\frac{1}{G_2\text{OIP3}_1} + \frac{1}{\text{OIP3}_2}\right]^{-1}, \text{ Coherent (Voltage)}$$

$$\text{OIP3} = \left[\frac{1}{G_2^2\text{OIP3}_1^2} + \frac{1}{\text{OIP3}_2^2}\right]^{-\frac{1}{2}}, \text{ Non-Coherent (Power)}$$

- $\text{OP1dB}^{\text{dBm}} = \text{OIP3}^{\text{dBm}} - 15.41^{\text{dB}}$

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Nonlinearity, Noise and Dynamic Range

White Noise

- White noise is any noise that has a flat frequency spectrum over the bandwidth of interest. The maximum **available noise power**, denoted by N, from any port is given by $N = kT_eB$ in which k is the Boltzmann constant, T_e is the port's **equivalent noise temperature** (in Kelvin) and B is the bandwidth (in Hertz). The equivalent noise temperature is obtained from the **noise figure** F which itself is determined through measurement.

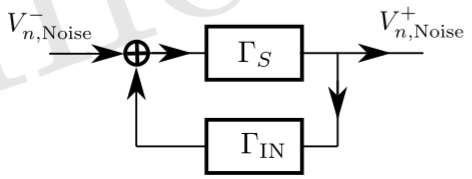
  The noise figure is defined as $F \equiv \frac{\left(\frac{S_{\text{IN}}}{N_0}\right)}{\left(\frac{S_{\text{OUT}}}{N_{\text{OUT}}}\right)}$ in which $\frac{S_{\text{IN}}}{N_0}$ is the **signal-to-noise-ratio** (SNR) at the input port, $\frac{S_{\text{OUT}}}{N_{\text{OUT}}}$ is the SNR at the output port and $N_0 = kT_0B$ in which $T_0 = 290^{(\circ K)}$.

- **Noise Characteristics of Two-Port Networks**
  - The Equivalent Temperature

    $T_e = (F-1)\,T_0$
  - The Noise Figure of a Cascade of Two-Ports

    $$F = F_1 + \frac{F_2 - 1}{G_{A,1}} + \frac{F_3 - 1}{G_{A,1}G_{A,2}} + \cdots$$
  - The Positive Feedback Loop of Noise at the Ports

    $$\frac{V^+_{n,\text{Noise}}}{V^-_{n,\text{Noise}}} = \frac{\Gamma_S}{1 - \Gamma_S\Gamma_{\text{IN}}}$$



- **An Illustrative Example:**
  The Equivalent Temperature of a Low-loss Transmission Line

  $$T_e = (L-1)\left(\frac{1 + \frac{1}{L}\left|\Gamma_S\right|^2}{1 - \left|\Gamma_S\right|^2}\right)T,\ L \equiv e^{2\alpha l}$$

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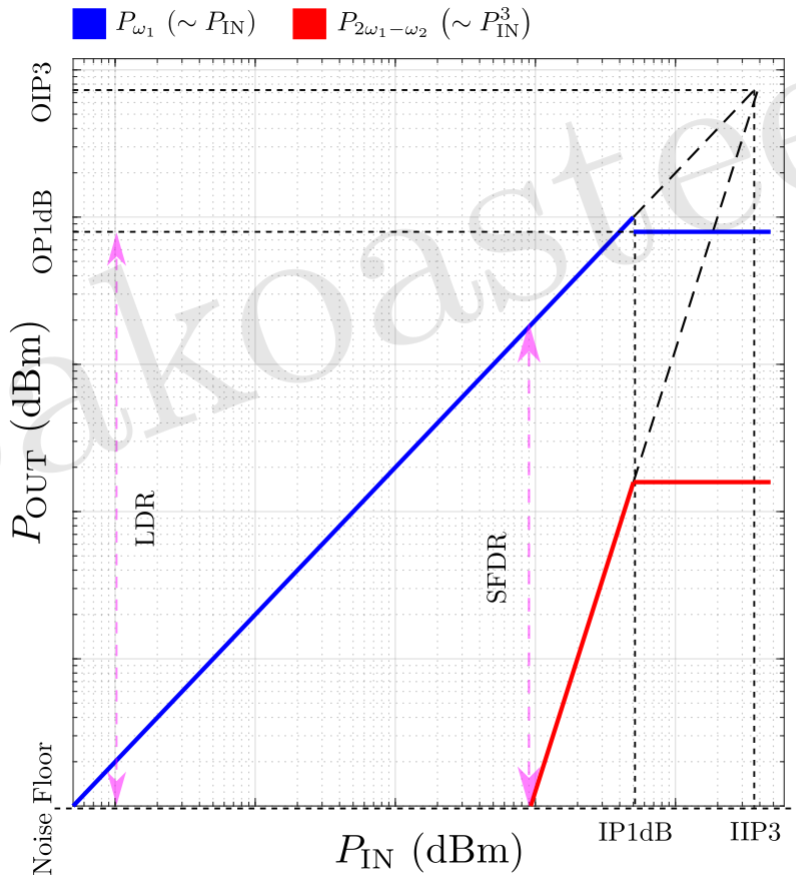
Nonlinearity, Noise and Dynamic Range

Dynamic Range

- The **linear dynamic range** (LDR) is the power range between the noise floor N and the P1dB. Therefore, $\text{LDR}^{\text{dB}} = \text{OP1dB}^{\text{dBm}} - N^{\text{dBm}}$. This definition is suitable for **power amplifiers**.

- The **spurious-free dynamic range** (SFDR) is the power range between the noise floor N and that level of P_{ω_1} at which $P_{2\omega_1-\omega_2} = N$. Therefore, $\text{SFDR}^{\text{dB}} = \frac{2}{3}\left(\text{OIP3}^{\text{dBm}} - N^{\text{dBm}}\right)$. This definition is suitable for **low-noise amplifiers** and **mixers**.



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Canonical Problems in Electrodynamics Radiating Elements and Arrays

Hypothetical Source Distributions

■ **Impulse of Electric Current**

$$\boldsymbol{J}(\boldsymbol{r}) = I_0\delta(x)\,\delta(y)\,\delta(z)\,\boldsymbol{a}_z$$

$$N_\theta(\boldsymbol{r}) = -I_0\sin\theta$$

$$N_\phi(\boldsymbol{r}) = 0$$

■ **Linear Constant Electric Current**

$$\boldsymbol{J}(\boldsymbol{r}) = I_0\delta(x)\,\delta(y)\,\Pi\left(\frac{z}{l}\right)\boldsymbol{a}_z$$

$$N_\theta(\boldsymbol{r}) = -(lI_0)\left[\frac{\sin\left(\frac{kl}{2}\cos\theta\right)}{\frac{kl}{2}\cos\theta}\right]\sin\theta$$

$$N_\phi(\boldsymbol{r}) = 0$$

■ **Linear Sinusoidal Electric Current**

$$\boldsymbol{J}(\boldsymbol{r}) = I_0\delta(x)\,\delta(y)\sin\left(k\left(\frac{l}{2}-|z|\right)\right)\boldsymbol{a}_z,\ |z|\le\frac{l}{2}$$

$$N_\theta(\boldsymbol{r}) = -\frac{2}{k}I_0\left(\frac{\cos\left(\frac{kl}{2}\cos\theta\right)-\cos\left(\frac{kl}{2}\right)}{\sin^2\theta}\right)\sin\theta$$

$$N_\phi(\boldsymbol{r}) = 0$$

■ **Circular (Loop) Constant Electric Current**

$$\boldsymbol{J}(\boldsymbol{r}) = I_0\delta(\rho-\rho_0)\,\delta(z)\,\boldsymbol{a}_\phi$$

$$N_\theta(\boldsymbol{r}) = 0$$

$$N_\phi(\boldsymbol{r}) = j\left([2\pi\rho_0]\,I_0\right)J_1(k\rho_0\sin\theta)$$

■ **Impulse of Magnetic Current**

$$\boldsymbol{M}(\boldsymbol{r}) = I_0^m\delta(x)\,\delta(y)\,\delta(z)\,\boldsymbol{a}_z$$

$$L_\theta(\boldsymbol{r}) = -I_0^m\sin\theta$$

$$L_\phi(\boldsymbol{r}) = 0$$

■ **Linear Constant Magnetic Current**

$$\boldsymbol{M}(\boldsymbol{r}) = I_0^m\delta(x)\,\delta(y)\,\Pi\left(\frac{z}{l}\right)\boldsymbol{a}_z$$

$$L_\theta(\boldsymbol{r}) = -(lI_0^m)\left[\frac{\sin\left(\frac{kl}{2}\cos\theta\right)}{\frac{kl}{2}\cos\theta}\right]\sin\theta$$

$$L_\phi(\boldsymbol{r}) = 0$$

■ **Circular (Loop) Sinusoidal Magnetic Current**

$$\boldsymbol{M}(\boldsymbol{r}) = I_0^m\delta(\rho-\rho_0)\cos(\phi)\,\delta(z)\,\boldsymbol{a}_\phi$$

$$L_\theta(\boldsymbol{r}) = -\left([2\pi\rho_0]\,I_0^m\right)\left[\frac{J_1(k\rho_0\sin\theta)}{k\rho_0\sin\theta}\right]\cos\theta\sin\phi$$

$$L_\phi(\boldsymbol{r}) = \left([2\pi\rho_0]\,I_0^m\right)J_1'(k\rho_0\sin\theta)\cos\phi$$

■ **Uniformly Illuminated Rectangular Aperture**

$$\boldsymbol{E}(\boldsymbol{r}) = E_0\Pi\left(\frac{x}{a}\right)\Pi\left(\frac{y}{b}\right)\delta(z)\,\boldsymbol{a}_y$$

$$\boldsymbol{M}(\boldsymbol{r}) = -2\boldsymbol{n}\times\boldsymbol{E}(\boldsymbol{r}) = I_0^m\Pi\left(\frac{x}{a}\right)\Pi\left(\frac{y}{b}\right)\delta(z)\,\boldsymbol{a}_x$$

$$L_\theta(\boldsymbol{r}) = \left([ab]\,I_0^m\right)\left[\frac{\sin\left(\frac{ka}{2}\sin\theta\cos\phi\right)}{\frac{ka}{2}\sin\theta\cos\phi}\right]\left[\frac{\sin\left(\frac{kb}{2}\sin\theta\sin\phi\right)}{\frac{kb}{2}\sin\theta\sin\phi}\right]\cos\theta\cos\phi$$

$$L_\phi(\boldsymbol{r}) = -\left([ab]\,I_0^m\right)\left[\frac{\sin\left(\frac{ka}{2}\sin\theta\cos\phi\right)}{\frac{ka}{2}\sin\theta\cos\phi}\right]\left[\frac{\sin\left(\frac{kb}{2}\sin\theta\sin\phi\right)}{\frac{kb}{2}\sin\theta\sin\phi}\right]\sin\phi$$

■ **Uniformly Illuminated Circular Aperture**

$$\boldsymbol{E}(\boldsymbol{r}) = E_0\delta(z)\,\boldsymbol{a}_y,\ \ \rho\le\rho_0$$

$$\boldsymbol{M}(\boldsymbol{r}) = -2\boldsymbol{n}\times\boldsymbol{E}(\boldsymbol{r}) = I_0^m\delta(z)\,\boldsymbol{a}_x,\ \ \rho\le\rho_0$$

$$L_\theta(\boldsymbol{r}) = 2\left(\left[\pi\rho_0^2\right]I_0^m\right)\left[\frac{J_1(k\rho_0\sin\theta)}{k\rho_0\sin\theta}\right]\cos\theta\cos\phi$$

$$L_\phi(\boldsymbol{r}) = -2\left(\left[\pi\rho_0^2\right]I_0^m\right)\left[\frac{J_1(k\rho_0\sin\theta)}{k\rho_0\sin\theta}\right]\sin\phi$$

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Canonical Problems in Electrodynamics
Radiating Elements and Arrays

The Array Factor

When the source distribution comprises multiple spatially-translated and scaled copies of a basic distribution, its radiation pattern can be expressed as the **multiplication** of the radiation pattern of that basic distribution with a quantity which is referred to as the **Array Factor**. For translations along a single axis (i.e., one-dimensional translations), this quantity is given by $\mathrm{AF}_N \equiv \sum_{n=1}^{N} A_n e^{jku_n \boldsymbol{a}_r \cdot \boldsymbol{a}_u}$ (the symbol u stands for any of the rectangular coordinates x, y or z).

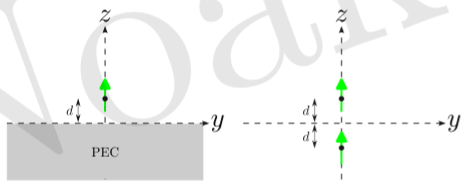
This procedure can be justified by considering the simple case in which the basic distribution is an impulse. Consider the source distribution given by $\boldsymbol{J}(\boldsymbol{r}) = \sum_{n=1}^{N} A_n \delta(x)\delta(y)\delta(z - z_n)\,\boldsymbol{a}_v$. This distribution describes impulses of electric current located along the z axis and directed along the v axis (the symbol v stands for any of the rectangular coordinates x, y or z). This distribution can be expressed as the convolution of a basic distribution, i.e., $\delta(x)\delta(y)\delta(z)\boldsymbol{a}_v$, with an impulse train, i.e., $\sum_{n=1}^{N} A_n \delta(x)\delta(y)\delta(z - z_n)$; In other words, $\boldsymbol{J}(\boldsymbol{r}) = [\delta(x)\delta(y)\delta(z)\boldsymbol{a}_v] * \left[\sum_{n=1}^{N} A_n \delta(x)\delta(y)\delta(z - z_n)\right]$. The radiation pattern is obtained using the far-field radiation integral $\boldsymbol{N}(\boldsymbol{r})$ which itself is a three-dimensional inverse Fourier transform. The Fourier transform converts convolution in one domain into multiplication in the other. Therefore, the radiation pattern for $\boldsymbol{J}(\boldsymbol{r})$ can be expressed as the multiplication of the radiation pattern for $\delta(x)\delta(y)\delta(z)\boldsymbol{a}_v$ and the inverse Fourier transform of $\sum_{n=1}^{N} A_n \delta(x)\delta(y)\delta(z - z_n)$. The latter quantity is referred to as the Array Factor and is given by $\mathrm{AF}_N \equiv \sum_{n=1}^{N} A_n e^{jkz_n \boldsymbol{a}_r \cdot \boldsymbol{a}_z}$.

The Image Theory

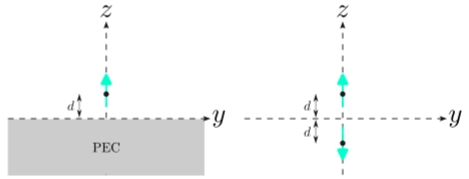
When sources $\boldsymbol{J}$ and $\boldsymbol{M}$ radiate in the presence of a ground plane (which coincides with the planar surface S), the radiation integrals $\boldsymbol{N}$ and $\boldsymbol{L}$ can't be used to obtain the radiated fields. To use $\boldsymbol{N}$ and $\boldsymbol{L}$, an equivalent configuration is set up in which (1) the ground plane is removed and (2) an imaginary (fictitious) source is placed on the other side of S such that the total electric field over S has no tangential component and, also, that the normal component of the total electric field is continuous across S.

The radiated fields obtained using this equivalent configuration are only valid for that side of S which contains the original source.

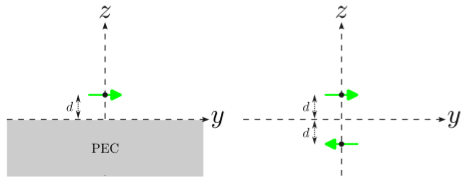
Vertical Electric Current



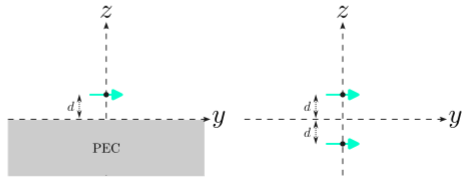
Vertical Magnetic Current



Horizontal Electric Current



Horizontal Magnetic Current



Oamed Nakoasteen

Canonical Problems in Electrodynamics
Radiating Elements and Arrays

Surface (Field) Equivalence

The Original Configuration

Sources $\boldsymbol{J}$ and $\boldsymbol{M}$ are radiating the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ everywhere in space. These sources and, by extension, the fields that they radiate are unknow. However, the tangential components of $\boldsymbol{E}$ and $\boldsymbol{H}$ are known over a segment S_k of a closed surface S that encloses the unknown sources. The problem is to find the radiated fields $\boldsymbol{E}$ and $\boldsymbol{H}$ outside the closed surface S.

The Equivalent Configuration

To find the fields radiated by the unknown sources $\boldsymbol{J}$ and $\boldsymbol{M}$ outside of the closed surface S, an equivalent configuration is set up in which (1) the unknown sources don't exist, (2) the total fields inside the surface are zero (Love's Equivalence Theorem) and (3) one of the two the surface current densities, i.e., $\boldsymbol{J}_s$ or $\boldsymbol{M}_s$, is placed over surface S which will radiate $\boldsymbol{E}$ and $\boldsymbol{H}$ (i.e., the same fields that are radiated by the unknown sources) outside of the surface S.

The **uniqueness theorem** states that, the fields outside a closed surface in a source-less region are uniquely determined by the tangential components of either the electric field or the magnetic field over that surface. Therefore, in the equivalent configuration, if $\boldsymbol{J}_s$ or $\boldsymbol{M}_s$ give the same tangential field components on S as the original configuration, then they will radiate the same fields, i.e., $\boldsymbol{E}$ and $\boldsymbol{H}$ outside of surface S.

In the equivalent configuration, the volume enclosed by the surface S is filled with PEC, $\boldsymbol{J}_s = \boldsymbol{0}$ and the magnetic surface current density $\boldsymbol{M}_s$ is placed on S such that (1) the tangential component of the electric field over the segment S_k is the same as the original configuration; In other words, $\boldsymbol{M}_s = -\boldsymbol{n} \times \boldsymbol{E}$ over the segment S_k and (2) it is zero everywhere else, i.e., over $S - S_k$. Since $\boldsymbol{M}_s$ is zero over $S - S_k$, it follows that the tangential component of the electric field is zero over $S - S_k$. Therefore, the closed surface S must be defined in such a way that, in the original configuration, the segment $S - S_k$ coincides with those surfaces over which the tangential component of the electric field is zero. In other words, in the original configuration, $S - S_k$ must coincide with PEC surfaces. Since in the equivalent configuration the tangential component of the electric field over the closed surface S is the same as the original configuration, it follows from the uniqueness theorem that the fields radiated by $\boldsymbol{M}_s$ outside of S must the same $\boldsymbol{E}$ and $\boldsymbol{H}$ as in the original configuration.

To use the radiation integrals to compute the fields radiated by $\boldsymbol{M}_s$, the **image theory** is used to remove the PEC from the equivalent configuration. Therefore, the final form for the magnetic current density which can be used in the radiation integrals is: $\boldsymbol{M}_s = -2\boldsymbol{n} \times \boldsymbol{E}$.

Physical Equivalence

The Original Configuration

Known incident fields $\boldsymbol{E}_{\text{inc}}$ and $\boldsymbol{H}_{\text{inc}}$ impinge on a PEC object and as a result, unknown current densities $\boldsymbol{J}_s$ and $\boldsymbol{M}_s$ form on its surface. These unknown surface current densities radiate unknown scattered fields $\boldsymbol{E}_S$ and $\boldsymbol{H}_S$ outside of the PEC object. The total fields outside of the PEC object are, therefore, given by $\boldsymbol{E} = \boldsymbol{E}_{\text{inc}} + \boldsymbol{E}_S$ and $\boldsymbol{H} = \boldsymbol{H}_{\text{inc}} + \boldsymbol{H}_S$ while $\boldsymbol{M}_s = -\boldsymbol{n} \times \boldsymbol{E}$ and $\boldsymbol{J}_s = \boldsymbol{n} \times \boldsymbol{H}$. The radiation integrals must be used to find the unknown surface current densities and the scattered fields that they radiate. The problem, however, is that they can only be used for current densities that radiate in a uniform region of space while the surface current densities in this situation are radiating in the presence of the PEC object.

The Equivalent Configuration

An equivalent configuration is set up in which (1) the incident fields $\boldsymbol{E}_{\text{inc}}$ and $\boldsymbol{H}_{\text{inc}}$ are removed, (2) the PEC object is removed and (3) unknown surface current densities $\boldsymbol{J}_s$ and $\boldsymbol{M}_s$ are placed over the imaginary surface S that coincides with the boundary of the PEC object. Now, radiation integrals can be used. However, the fields that are calculated using these integrals will be valid only outside of the closed surface S; This is because, $\boldsymbol{M}_s = -\boldsymbol{n} \times [\boldsymbol{E}_S - (-\boldsymbol{E}_{\text{inc}})]$ and $\boldsymbol{J}_s = \boldsymbol{n} \times [\boldsymbol{H}_S - (-\boldsymbol{H}_{\text{inc}})]$ which means that outside of the closed surface S, these current densities will radiate $\boldsymbol{E}_S$ and $\boldsymbol{H}_S$ while inside of it they will radiate $-\boldsymbol{E}_{\text{inc}}$ and $-\boldsymbol{H}_{\text{inc}}$.

The **uniqueness theorem** states that, the fields outside a closed surface in a source-less region are uniquely determined by the tangential components of either the electric field or the magnetic field over that surface. Therefore, in the equivalent configuration, if either $\boldsymbol{M}_s$ or $\boldsymbol{J}_s$ is completely known over S, then the fields can be determined outside of S.

In the equivalent configuration, $\boldsymbol{M}_s = \boldsymbol{0}$; And the equations that can be solved to obtain $\boldsymbol{J}_s$ over S are $\boldsymbol{n} \times \boldsymbol{E}_S = -\boldsymbol{n} \times \boldsymbol{E}_{\text{inc}}$ and $\boldsymbol{n} \times \boldsymbol{H}_S - \boldsymbol{J}_s = -\boldsymbol{n} \times \boldsymbol{H}_{\text{inc}}$.

Canceled Material

Canonical Problems in Electrodynamics
Waveguides

Rectangular Waveguides: TM^z_{mn}

$$E_x^+ = -j\frac{k_x k_z}{k_c^2}E_0^+ \cos\left(m\frac{x}{a}\pi\right)\sin\left(n\frac{y}{b}\pi\right)e^{-jk\sqrt{1-\left(\frac{f_c^{mn}}{f}\right)^2}z}$$

$$E_y^+ = -j\frac{k_y k_z}{k_c^2}E_0^+ \sin\left(m\frac{x}{a}\pi\right)\cos\left(n\frac{y}{b}\pi\right)e^{-jk\sqrt{1-\left(\frac{f_c^{mn}}{f}\right)^2}z}$$

$$E_z^+ = E_0^+ \sin\left(m\frac{x}{a}\pi\right)\sin\left(n\frac{y}{b}\pi\right)e^{-jk\sqrt{1-\left(\frac{f_c^{mn}}{f}\right)^2}z}$$

$$\boldsymbol{H}^+_{\mathrm{TM}^z_{mn}} = \frac{1}{Z_{\mathrm{TM}^z_{mn}}}\boldsymbol{a}_z \times \boldsymbol{E}^+_{\mathrm{TM}^z_{mn}}$$

$$\boldsymbol{P}_{\mathrm{av}} = \frac{1}{2Z_{\mathrm{TM}^z_{mn}}}\left(E_x^+E_x^{+*} + E_y^+E_y^{+*}\right)\boldsymbol{a}_z = \frac{Z_{\mathrm{TM}^z_{mn}}}{2}\left(H_x^+H_x^{+*} + H_y^+H_y^{+*}\right)\boldsymbol{a}_z$$

$$k_c^2 \equiv k_x^2 + k_y^2,\ f_c^{mn} = \frac{1}{2a\sqrt{\mu\epsilon}}\sqrt{m^2 + \left(\frac{a}{b}n\right)^2}$$

$$Z_{\mathrm{TM}^z_{mn}} \equiv \frac{E_x^+}{H_y^+} = -\frac{E_y^+}{H_x^+} = \frac{k_z}{\omega\epsilon} = \eta\sqrt{1-\left(\frac{f_c^{mn}}{f}\right)^2}$$

$$\frac{E_x^-}{E_x^+} = -\frac{E_0^-}{E_0^+}e^{j2k_z z} \quad \frac{H_x^-}{H_x^+} = \frac{E_0^-}{E_0^+}e^{j2k_z z}$$

$$\frac{E_y^-}{E_y^+} = -\frac{E_0^-}{E_0^+}e^{j2k_z z} \quad \frac{H_y^-}{H_y^+} = \frac{E_0^-}{E_0^+}e^{j2k_z z}$$

$$\frac{E_z^-}{E_z^+} = \frac{E_0^-}{E_0^+}e^{j2k_z z}$$

Rectangular Waveguides: TE^z_{mn}

$$H_x^+ = j\frac{k_x k_z}{k_c^2}H_0^+ \sin\left(m\frac{x}{a}\pi\right)\cos\left(n\frac{y}{b}\pi\right)e^{-jk\sqrt{1-\left(\frac{f_c^{mn}}{f}\right)^2}z}$$

$$H_y^+ = j\frac{k_y k_z}{k_c^2}H_0^+ \cos\left(m\frac{x}{a}\pi\right)\sin\left(n\frac{y}{b}\pi\right)e^{-jk\sqrt{1-\left(\frac{f_c^{mn}}{f}\right)^2}z}$$

$$H_z^+ = H_0^+ \cos\left(m\frac{x}{a}\pi\right)\cos\left(n\frac{y}{b}\pi\right)e^{-jk\sqrt{1-\left(\frac{f_c^{mn}}{f}\right)^2}z}$$

$$\boldsymbol{E}^+_{\mathrm{TE}^z_{mn}} = -Z_{\mathrm{TE}^z_{mn}}\boldsymbol{a}_z \times \boldsymbol{H}^+_{\mathrm{TE}^z_{mn}}$$

$$\boldsymbol{P}_{\mathrm{av}} = \frac{1}{2Z_{\mathrm{TE}^z_{mn}}}\left(E_x^+E_x^{+*} + E_y^+E_y^{+*}\right)\boldsymbol{a}_z = \frac{Z_{\mathrm{TE}^z_{mn}}}{2}\left(H_x^+H_x^{+*} + H_y^+H_y^{+*}\right)\boldsymbol{a}_z$$

$$k_c^2 \equiv k_x^2 + k_y^2,\ f_c^{mn} = \frac{1}{2a\sqrt{\mu\epsilon}}\sqrt{m^2 + \left(\frac{a}{b}n\right)^2}$$

$$Z_{\mathrm{TE}^z_{mn}} \equiv \frac{E_x^+}{H_y^+} = -\frac{E_y^+}{H_x^+} = \frac{\omega\mu}{k_z} = \frac{\eta}{\sqrt{1-\left(\frac{f_c^{mn}}{f}\right)^2}}$$

$$\frac{H_x^-}{H_x^+} = -\frac{H_0^-}{H_0^+}e^{j2k_z z} \quad \frac{E_x^-}{E_x^+} = \frac{H_0^-}{H_0^+}e^{j2k_z z}$$

$$\frac{H_y^-}{H_y^+} = -\frac{H_0^-}{H_0^+}e^{j2k_z z} \quad \frac{E_y^-}{E_y^+} = \frac{H_0^-}{H_0^+}e^{j2k_z z}$$

$$\frac{H_z^-}{H_z^+} = \frac{H_0^-}{H_0^+}e^{j2k_z z}$$

Oameed Noakoasteen

Canonical Problems in Electrodynamics
Waveguides

Circular Waveguides: TM^z_{mn}

- $$E^+_\rho = -j \quad \frac{k_z}{k_c} E^+_0 J'_m\left(\chi_{mn}\frac{\rho}{\rho_0}\right)\cos(m\phi)\, e^{-jk\sqrt{1-\left(\frac{f^{mn}_c}{f}\right)^2}z}$$
  $$E^+_\phi = \quad jm\frac{k_z}{k_c^2}E^+_0 \frac{J_m\left(\chi_{mn}\frac{\rho}{\rho_0}\right)}{\rho}\sin(m\phi)\, e^{-jk\sqrt{1-\left(\frac{f^{mn}_c}{f}\right)^2}z}$$
  $$E^+_z = \quad E^+_0 J_m\left(\chi_{mn}\frac{\rho}{\rho_0}\right)\cos(m\phi)\, e^{-jk\sqrt{1-\left(\frac{f^{mn}_c}{f}\right)^2}z}$$

$$\boldsymbol{H}^+_{\mathrm{TM}^z_{mn}} = \frac{1}{Z_{\mathrm{TM}^z_{mn}}}\boldsymbol{a}_z \times \boldsymbol{E}^+_{\mathrm{TM}^z_{mn}}$$

$$\boldsymbol{P}^+_{\mathrm{av}} = \frac{1}{2Z_{\mathrm{TM}^z_{mn}}}\left(E^+_\rho E^{+*}_\rho + E^+_\phi E^{+*}_\phi\right)\boldsymbol{a}_z = \frac{Z_{\mathrm{TM}^z_{mn}}}{2}\left(H^+_\rho H^{+*}_\rho + H^+_\phi H^{+*}_\phi\right)\boldsymbol{a}_z$$

- $$k_c^2 \equiv k_\rho^2,\ f^{mn}_c = \frac{1}{2\pi\rho_0\sqrt{\mu\epsilon}}\chi_{mn}$$
  $$Z_{\mathrm{TM}^z_{mn}} \equiv \frac{E^+_\rho}{H^+_\phi} = -\frac{E^+_\phi}{H^+_\rho} = \frac{k_z}{\omega\epsilon} = \eta\sqrt{1-\left(\frac{f^{mn}_c}{f}\right)^2}$$

- $$\frac{E^-_\rho}{E^+_\rho} = -\frac{E^-_0}{E^+_0}e^{j2k_z z} \qquad \frac{H^-_\rho}{H^+_\rho} = \frac{E^-_0}{E^+_0}e^{j2k_z z}$$
  $$\frac{E^-_\phi}{E^+_\phi} = -\frac{E^-_0}{E^+_0}e^{j2k_z z} \qquad \frac{H^-_\phi}{H^+_\phi} = \frac{E^-_0}{E^+_0}e^{j2k_z z}$$
  $$\frac{E^-_z}{E^+_z} = \frac{E^-_0}{E^+_0}e^{j2k_z z}$$

Circular Waveguides: TE^z_{mn}

- $$H^+_\rho = -j \quad \frac{k_z}{k_c} H^+_0 J'_m\left(\chi'_{mn}\frac{\rho}{\rho_0}\right)\cos(m\phi)\, e^{-jk\sqrt{1-\left(\frac{f^{mn}_c}{f}\right)^2}z}$$
  $$H^+_\phi = \quad jm\frac{k_z}{k_c^2}H^+_0 \frac{J_m\left(\chi'_{mn}\frac{\rho}{\rho_0}\right)}{\rho}\sin(m\phi)\, e^{-jk\sqrt{1-\left(\frac{f^{mn}_c}{f}\right)^2}z}$$
  $$H^+_z = \quad H^+_0 J_m\left(\chi'_{mn}\frac{\rho}{\rho_0}\right)\cos(m\phi)\, e^{-jk\sqrt{1-\left(\frac{f^{mn}_c}{f}\right)^2}z}$$

$$\boldsymbol{E}^+_{\mathrm{TE}^z_{mn}} = -Z_{\mathrm{TE}^z_{mn}}\boldsymbol{a}_z \times \boldsymbol{H}^+_{\mathrm{TE}^z_{mn}}$$

$$\boldsymbol{P}^+_{\mathrm{av}} = \frac{1}{2Z_{\mathrm{TE}^z_{mn}}}\left(E^+_\rho E^{+*}_\rho + E^+_\phi E^{+*}_\phi\right)\boldsymbol{a}_z = \frac{Z_{\mathrm{TE}^z_{mn}}}{2}\left(H^+_\rho H^{+*}_\rho + H^+_\phi H^{+*}_\phi\right)\boldsymbol{a}_z$$

- $$k_c^2 \equiv k_\rho^2,\ f^{mn}_c = \frac{1}{2\pi\rho_0\sqrt{\mu\epsilon}}\chi'_{mn}$$
  $$Z_{\mathrm{TE}^z_{mn}} \equiv \frac{E^+_\rho}{H^+_\phi} = -\frac{E^+_\phi}{H^+_\rho} = \frac{\omega\mu}{k_z} = \frac{\eta}{\sqrt{1-\left(\frac{f^{mn}_c}{f}\right)^2}}$$

- $$\frac{E^-_\rho}{E^+_\rho} = -\frac{E^-_0}{E^+_0}e^{j2k_z z} \qquad \frac{H^-_\rho}{H^+_\rho} = \frac{E^-_0}{E^+_0}e^{j2k_z z}$$
  $$\frac{E^-_\phi}{E^+_\phi} = -\frac{E^-_0}{E^+_0}e^{j2k_z z} \qquad \frac{H^-_\phi}{H^+_\phi} = \frac{E^-_0}{E^+_0}e^{j2k_z z}$$
  $$\frac{E^-_z}{E^+_z} = \frac{E^-_0}{E^+_0}e^{j2k_z z}$$

Oameed Noakoasteen

Canonical Problems in Electrodynamics
Propagation in Material Media

Lossy Dielectric Media

- In a lossy dielectric medium, the constitutive parameter is given by $\epsilon_c = \epsilon^{'}(1 - j\tan\delta)$ in which $\epsilon^{'} \equiv \epsilon_0\epsilon_r^{'}$ represents the **permittivity** of the medium and $\tan\delta \equiv \frac{\sigma_{\text{eff}}}{\omega\epsilon^{'}}$, referred to as the **loss tangent**, represents the losses. The **effective conductivity** is defined as $\sigma_{\text{eff}} \equiv \sigma + \omega\epsilon^{''}$ in which σ represents the **conduction (ohmic) loss** mechanism and $\epsilon^{''} \equiv \epsilon_0\epsilon_r^{''}$ represents the **dielectric loss** mechanism.

- The **phase velocity**, denoted by u_p, is defined as $u_p \equiv \lambda_g f$ in which λ_g is the wavelength of the propagating wave in the medium. Since $\lambda_g \equiv \frac{2\pi}{\beta}$, it follows that $u_p = \frac{\omega}{\beta}$. The **group velocity**, denoted by u_g, is given by $u_g = \left(\frac{\partial\beta}{\partial\omega}\right)^{-1}$. The **refractive index**, denoted by n, is defined as $n \equiv \frac{c_0}{u_p}$ in which c_0 is the speed of the wave in free space.

- **The Uniform Plane Wave**

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{-\gamma \boldsymbol{a}_{r,p}\cdot\boldsymbol{r}}$$

$$\boldsymbol{H} = \frac{1}{\eta_c}\boldsymbol{a}_{r,p} \times \boldsymbol{E}$$

  - The unit vector $\boldsymbol{a}_p$ represents the direction of propagation. The vector $\boldsymbol{E}_0 = E_{0x}\boldsymbol{a}_x + E_{0y}\boldsymbol{a}_y + E_{0z}\boldsymbol{a}_z$ represents the constant electric field on the plane that is transverse to $\boldsymbol{a}_{r,p}$. The vectors $\boldsymbol{a}_{r,p}$ and $\boldsymbol{E}_0$ must satisfy $\boldsymbol{a}_{r,p} \cdot \boldsymbol{E}_0 = 0$. The **propagation constant**, denoted by γ, must satisfy $\gamma^2 = -\omega^2\mu_0\epsilon_c$. The **complex wave impedance**, denoted by η_c, is given by $\eta_c = j\frac{\omega\mu_0}{\gamma}$.
  - The **attenuation constant**, denoted by α, is defined as $\alpha \equiv \text{Re}(\gamma)$; Therefore, $\alpha = \frac{1}{\sqrt{2}}\omega\sqrt{\mu_0\epsilon^{'}}\sqrt{-1 + \sqrt{1 + (\tan\delta)^2}}$.
    The **phase constant**, denoted by β, is defined as $\beta \equiv \text{Im}(\gamma)$. Therefore, $\beta = \frac{1}{\sqrt{2}}\omega\sqrt{\mu_0\epsilon^{'}}\sqrt{1 + \sqrt{1 + (\tan\delta)^2}}$.

- A medium for which $\tan\delta \ll 1$, is a **good dielectric**; In this case, $\alpha = \frac{1}{2}\sigma_{\text{eff}}\sqrt{\frac{\mu_0}{\epsilon^{'}}}$, $\beta = \omega\sqrt{\mu_0\epsilon^{'}}$ and $\eta_c = \sqrt{\frac{\mu_0}{\epsilon^{'}}}$.
  A medium for which $\tan\delta \gg 1$, is a **good conductor**; In this case, $\alpha = \sqrt{\frac{1}{2}\sigma_{\text{eff}}\omega\mu_0}$, $\beta = \sqrt{\frac{1}{2}\sigma_{\text{eff}}\omega\mu_0}$ and $\eta_c = \sqrt{\frac{1}{2}\sigma_{\text{eff}}^{-1}\omega\mu_0}\,(1 + j)$.

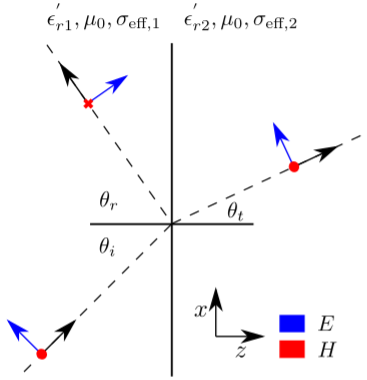
Oameed Noaesteen

Canonical Problems in Electrodynamics
Propagation in Material Media

Lossy Dielectric Media (Cont'd)

Reflection and Transmission at the Boundaries: The Dielectric-Dielectric Interface

The Parallel Mode



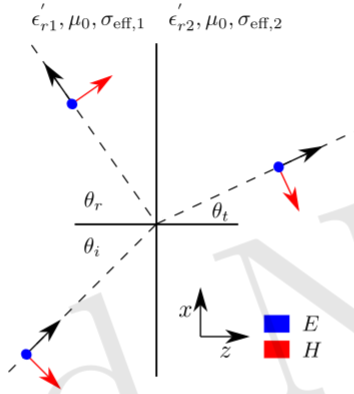
$$\theta_r = \theta_i$$

$$\sin\theta_t = \frac{\gamma_1}{\gamma_2}\sin\theta_i$$

$$\Gamma = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$$

$$T = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$$

The Perpendicular Mode



$$\theta_r = \theta_i$$

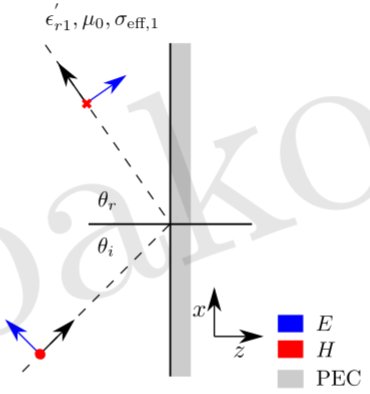
$$\sin\theta_t = \frac{\gamma_1}{\gamma_2}\sin\theta_i$$

$$\Gamma = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}$$

$$T = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}$$

Reflection and Transmission at the Boundaries: The Dielectric-PEC Interface

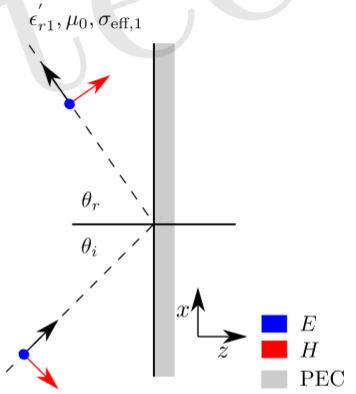
The Parallel Mode



$$\theta_r = \theta_i$$

$$\Gamma = -1$$

The Perpendicular Mode



$$\theta_r = \theta_i$$

$$\Gamma = -1$$

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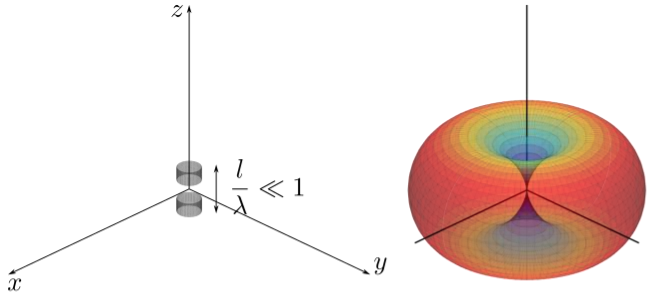
Microwave Circuits
Antenna Elements and Arrays

■ Infinitesimal Dipole



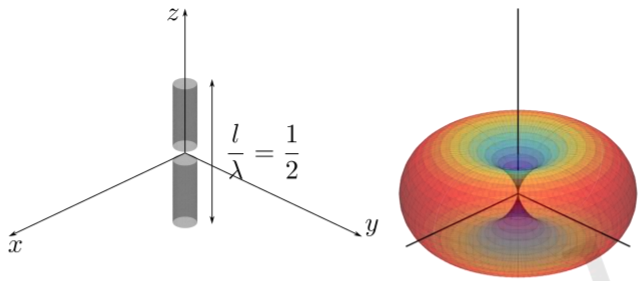
- $\boldsymbol{J}(\boldsymbol{r}) = I_0\delta(x)\delta(y)\Pi\left(\frac{z}{l}\right)\boldsymbol{a}_z$
- $E_r = 0$

  $E_\theta = j\frac{\eta}{2}\left(\frac{l}{\lambda}\right)I_0\frac{e^{-jkr}}{r}\sin\theta$

  $E_\phi = 0$

  $\boldsymbol{H} = \frac{1}{\eta}\boldsymbol{a}_r \times \boldsymbol{E}$

■ Half-Wave Dipole



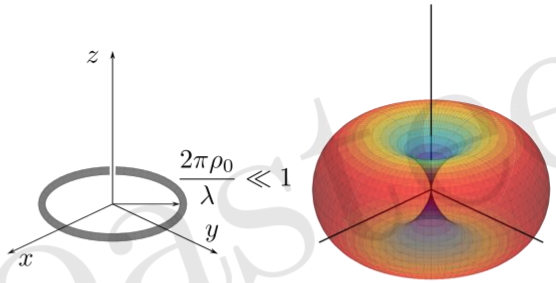
- $\boldsymbol{J}(\boldsymbol{r}) = I_0\delta(x)\delta(y)\sin\left(\frac{\pi}{2}\left(1 - \frac{|z|}{\left(\frac{\lambda}{4}\right)}\right)\right)\boldsymbol{a}_z$
- $E_r = 0$

  $E_\theta = j\frac{\eta}{2\pi}I_0\frac{e^{-jkr}}{r}\left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right]$

  $E_\phi = 0$

  $\boldsymbol{H} = \frac{1}{\eta}\boldsymbol{a}_r \times \boldsymbol{E}$

■ Small Circular Loop



- $\boldsymbol{J}(\boldsymbol{r}) = I_0\delta(\rho - \rho_0)\delta(z)\boldsymbol{a}_\phi$
- $E_r = 0$

  $E_\theta = 0$

  $E_\phi = \frac{\eta}{4}\left(\frac{2\pi\rho_0}{\lambda}\right)^2 I_0\frac{e^{-jkr}}{r}\sin\theta$

  $\boldsymbol{H} = \frac{1}{\eta}\boldsymbol{a}_r \times \boldsymbol{E}$

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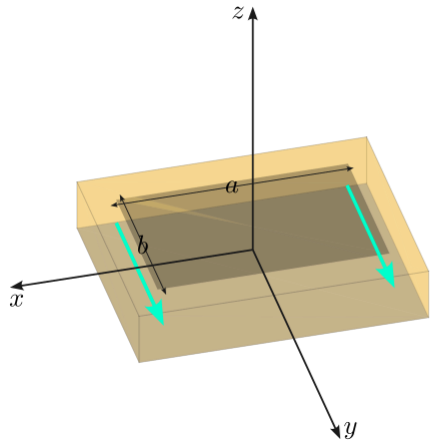
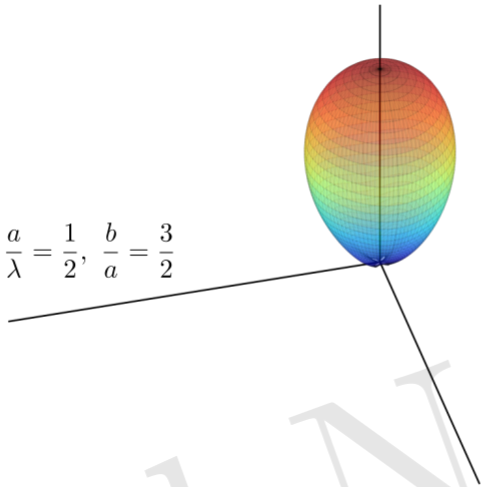
Microwave Circuits
Antenna Elements and Arrays

Rectangular Microstrip Patch





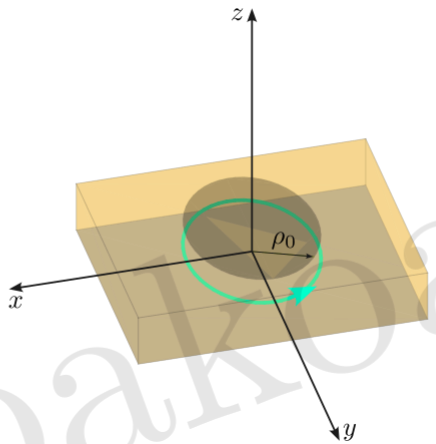
- $\boldsymbol{M}^{\mathrm{TM}^{z}_{10}}(\boldsymbol{r}) = j2\omega A_0 \delta(x)\Pi\left(\frac{y}{b}\right)\Pi\left(\frac{z}{h}\right)\boldsymbol{a}_y$
- $\mathrm{AF}_2 = 2\cos\left(\frac{ka}{2}\sin\theta\cos\phi\right)$
- $E_r = 0$

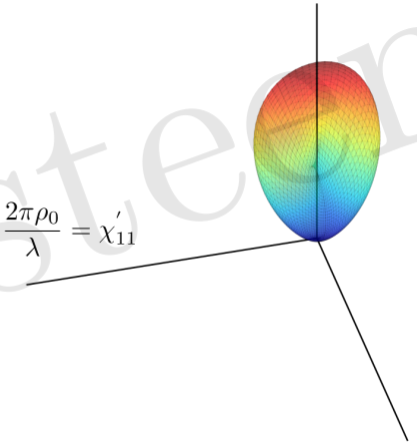
$$E_\theta = 4\pi c_0\left(\frac{bh}{\lambda^2}\right)A_0\frac{e^{-jkr}}{r}\left[\frac{\sin\left(\frac{kb}{2}\sin\theta\sin\phi\right)}{\frac{kb}{2}\sin\theta\sin\phi}\right]\cos\left(\frac{ka}{2}\sin\theta\cos\phi\right)\cos\phi$$

$$E_\phi = -4\pi c_0\left(\frac{bh}{\lambda^2}\right)A_0\frac{e^{-jkr}}{r}\left[\frac{\sin\left(\frac{kb}{2}\sin\theta\sin\phi\right)}{\frac{kb}{2}\sin\theta\sin\phi}\right]\cos\left(\frac{ka}{2}\sin\theta\cos\phi\right)\cos\theta\sin\phi$$

$$\boldsymbol{H} = \frac{1}{\eta}\boldsymbol{a}_r \times \boldsymbol{E}$$

Circular Microstrip Patch





- $\boldsymbol{M}^{\mathrm{TM}^{z}_{11}}(\boldsymbol{r}) = -j2\omega J_1\left(\chi'_{11}\right)A_0\delta(\rho-\rho_0)\cos(\phi)\,\delta(z)\boldsymbol{a}_\phi$
- $E_r = 0$

$$E_\theta = -2\pi c_0\left(\frac{J_1\left(\chi'_{11}\right)}{\lambda}\right)\left(\frac{2\pi\rho_0}{\lambda}\right)A_0\frac{e^{-jkr}}{r}J'_1\left(k\rho_0\sin\theta\right)\cos\phi$$

$$E_\phi = -2\pi c_0\left(\frac{J_1\left(\chi'_{11}\right)}{\lambda}\right)\left(\frac{2\pi\rho_0}{\lambda}\right)A_0\frac{e^{-jkr}}{r}\left[\frac{J_1\left(k\rho_0\sin\theta\right)}{k\rho_0\sin\theta}\right]\cos\theta\sin\phi$$

$$\boldsymbol{H} = \frac{1}{\eta}\boldsymbol{a}_r \times \boldsymbol{E}$$

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Microwave Circuits
Antenna Elements and Arrays

The Linear Array: Uniform Amplitude Elements

Antenna elements are placed along the z axis; Therefore, $\mathrm{AF}_N = \sum_{n=1}^{N} A_n e^{jkz_n \boldsymbol{a}_r \cdot \boldsymbol{a}_z}$. The first element is placed at the origin. The second element is placed a distance d away from the first one and so on; Therefore, $z_n = (n-1)\,d$. For all elements, $|A_n| = 1$. For the first element, $\phi_{A_1} = 0$. The second element has a phase shift β relative to the first one and so on; Therefore, $\phi_{A_n} = (n-1)\,\beta$. With this configuration, the array factor reduces to $\mathrm{AF}_N = \sum_{n'=0}^{N-1} e^{jn'\psi}$ in which $\psi \equiv kd\cos\theta + \beta$.

This expression can be reformulated (and normalized) as $\mathrm{AF}_n = \frac{\sin\left(N\frac{\psi}{2}\right)}{N\sin\left(\frac{\psi}{2}\right)}$ which, in turn, can be approximated as $\mathrm{AF}_N \approx \frac{\sin\left(N\frac{\psi}{2}\right)}{N\frac{\psi}{2}}$ for $\psi \ll 1$.

- Broadside Pattern

  Given that the $\max(\mathrm{AF}_N)$ occurs when $\psi = 0$, it follows that for AF_N to have its maximum at $\theta = \frac{\pi}{2}$, the phase shift must satisfy $\beta = 0$.

- Endfire Pattern

  Given that the $\max(\mathrm{AF}_N)$ occurs when $\psi = 0$, it follows that for AF_N to have its maximum at $\theta = 0$, the phase shift must satisfy $\beta = -kd$.

- Endfire Pattern: The Hansen-Woodyard Design

$$d = \left(\frac{N-1}{N}\right)\frac{\lambda}{4},\ \beta = -\left(\frac{N+1}{N}\right)\frac{\pi}{2}$$

The Linear Array: Non-Uniform Amplitude Elements

Antenna elements are placed along the z axis; Therefore, $\mathrm{AF}_N = \sum_{n=1}^{N} A_n e^{jkz_n \boldsymbol{a}_r \cdot \boldsymbol{a}_z}$. The elements are placed symmetrically about the origin and their amplitudes are required to be symmetrical with respect to the origin. For all elements $\phi_{A_n} = 0$.
With this configuration, the array factor reduces to $\mathrm{AF}_N = t + 2\sum_{n'=1}^{M} A_{n'} \cos\left(\gamma\frac{\psi}{2}\right)$ in which $\psi = kd\cos\theta$ and $M = \frac{N-1}{2}$, $t = A_{\frac{N+1}{2}}$, $\gamma = 2n'$ when $N \in \mathrm{O}$ and $M = \frac{N}{2}$, $t = 0$, $\gamma = 2n' - 1$ when $N \in \mathrm{E}$.

- Broadside Pattern: The Maximally Flat Design

  The $A_{n''}$ are determined by requiring $\mathrm{AF}_N = \left(2\cos\left(\frac{\psi}{2}\right)\right)^{N-1}$.

- Broadside Pattern: The Chebyshev Design

  The $A_{n''}$ are determined by requiring $\mathrm{AF}_N = \mathrm{T}_{N-1}\left(z_0\cos\left(\frac{\psi}{2}\right)\right)$ in which $z_0 = \cosh\left(\frac{1}{N-1}\cosh^{-1}(R_0)\right)$.
  The ratio of the intensity of the major lobe to that of the minor lobe, denoted by R_0, is ususally specified in dBs, i.e., $R_0^{\mathrm{dB}} = 20\log_{10}(R_0)$.

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Microwave Circuits
Antenna Elements and Arrays

A Medley of Definitions and Figures of Merit

- A **radiation lobe** is a portion of the radiation pattern bounded by regions of relatively weak radiation intensity. The **major lobe** (also referred to as the *main beam*) is the radiation lobe containing the direction of maximum radiation. A **minor lobe** is any lobe except the major lobe and represents radiation in undesired directions. The **side sobe** is usually the largest of the minor lobes and is adjacent to the main beam. The **back lobe** is a radiation lobe whose axis makes an angle of approximately 180° with respect to the major lobe.
- The **principal patterns** refer to the radiation pattern sampled over the two principle cut-planes, namely, the **E-plane** and the **H-plain**. The E-plane is defined as the plane containing the electric field vector and the direction of maximum radiation. The H-plane is defined as the plane containing the magnetic field vector and the direction of maximum radiation. Usually, antennas are oriented so that at least one of the principal planes coincides with either the $\phi = 0$ plane (i.e., the x-z plane, also referred to as the elevation plane) or the $\theta = \frac{\pi}{2}$ plane (i.e., the x-y plane, also referred to as the azimuthal plane).
- For a **pencil beam** pattern, the major lobe is relatively narrow in both principal planes (elevation and azimuthal). For an **omnidirectional** pattern, the major lobe is constant in the azimuthal plane.
- The **directivity** is defined as the ratio of the maximum of the radiation intensity to its average, i.e., $D_0 \equiv \frac{U_0}{U_{\text{av}}}$ in which $U_0 \equiv \max\left(U\left(\theta,\phi\right)\right)$ and $U_{\text{av}} = \frac{P_{\text{rad}}}{4\pi}$. The radiated power is given by $\iint_\Omega U\, d\Omega$ in which $d\Omega = \sin\theta d\theta d\phi$. Furthermore, $D_0^{\text{dB}} = 10\log\left(D_0\right)$.
  The directivity can also be expressed as $D_0 = \frac{4\pi}{\Omega_A}$ in which $\Omega_A \equiv \frac{P_{\text{rad}}}{U_0}$ and is referred to as the **beam solid angle**. The Ω_A can be interpreted as that solid angle through which all the power of the radiating source would flow if its radiation intensity were constant and equal to U_0 for all angles within Ω_A.

- The **gain**, is defined as $G_0 \equiv e_0 D_0$ in which e_0 is the **antenna efficiency**. Furthermore, $G_0^{\text{dB}} = 10\log G_0$.
- The **beam width** refers to the angular separation between two identical points on opposite sides of the pattern maximum. The **half-power beam width** (HPBW) refers to the angle between the two directions (in the plane containing the direction of the maximum of the beam) along which the radiation intensity is half the value of the beam maximum. The **first-null beam width** (FNBW) refers to the angular separation between the first nulls of the pattern.
- The **side lobe level** (SLL) is the ratio of the power density in the side lobe to that of the main beam.

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Microwave Circuits
Antenna Elements and Arrays

- **A Medley of Definitions and Figures of Merit (Cont'd)**
  - The **input impedance** is defined as the impedance presented by an antenna at its terminals. The real part of the input impedance, denoted by R_{IN}, represents power loss due to both radiation and (ohmic/dielectric) dissipation and is given by $R_{\text{IN}} = R_r + R_L$ in which R_r denotes the **radiation resistance** and R_L denotes the loss resistance. The radiation resistance is given by $R_r \equiv \frac{2P_{\text{rad}}}{|I_{\text{IN}}|^2}$.
  - The **vector effective length**, denoted by $\boldsymbol{l}_e$, (also referred to as the effective height) is used to determine the voltage induced on the open-circuit terminals of an antenna when a wave impinges upon it. It is defined as $\boldsymbol{l}_e = j\frac{1}{|I_{\text{IN}}|}\frac{4\pi r}{k\eta}\boldsymbol{E}_{\text{antenna}}$ and the open-circuit voltage is given by $V_{\text{oc}} = \boldsymbol{E}_{\text{wave}} \cdot \boldsymbol{l}_e$.
  - The **polarization** of an antenna is defined as the polarization of the wave radiated (transmitted) by that antenna.
  - The **polarization loss factor** (PLF) measures the polarization mismatch between the receiving antenna and the incident wave; It is given by $\text{PLF}^{\text{dB}} = 10\log\left(|\boldsymbol{a}_{\text{wave}} \cdot \boldsymbol{a}_{\text{antenna}}|^2\right)$.
  - The **maximum effective aperture**, denoted by A_{em}, is defined as the ratio of the available power at the terminals of a receiving antenna to the power density of a plane wave (with matching polarization) incident on that antenna (from the direction of the maximum of its radiation pattern); It is given by $A_{\text{em}} = \frac{\lambda^2}{4\pi}G_0$.
  - The Friis Transmission Formula

$$\frac{p_r}{p_t} = \left(\frac{\lambda}{4\pi r}\right)^2 G_{0t}G_{0r}$$

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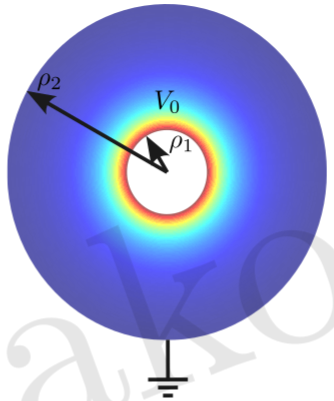
Microwave Circuits
Guiding Structures

Coaxial Transmission Line

- The **dominant mode** of propagation is TEM^z. The guided wavelength and the wave impedance are given by $\lambda_g = \frac{\lambda_0}{\sqrt{\epsilon_r}}$ and $Z_{\text{TEM}^z} = \frac{\eta_0}{\sqrt{\epsilon_r}}$.
- The cutoff frequency of the **next higher-order mode** can be found from the minimum k_ρ that satisfies either $J_m(k_\rho\rho_1)\,Y_m(k_\rho\rho_2) - Y_m(k_\rho\rho_1)\,J_m(k_\rho\rho_2) = 0$ (for the TM^z_{mn} modes) or $J_m^{'}(k_\rho\rho_1)\,Y_m^{'}(k_\rho\rho_2) - Y_m^{'}(k_\rho\rho_1)\,J_m^{'}(k_\rho\rho_2) = 0$ (for the TE^z_{mn} modes).
- From the transmission line model, for a lossless transmission line $Z_0 = \frac{2P_{\text{av}}}{|I^+|^2}$ (or equivalently, $Z_0 = \frac{|V^+|^2}{2P_{\text{av}}}$). Since $I^+ = \int_0^{2\pi} J_z(\rho=\rho_1,\phi,z=0)\,\rho_1 d\phi$ in which $J_z = -H_\phi$, it follows that the **characteristic impedance** for the coaxial transmission line is given by $Z_0 = \frac{1}{2\pi}\ln\left(\frac{\rho_2}{\rho_1}\right) Z_{\text{TEM}^z}$.



$$E_\rho^+ = \frac{1}{\ln\left(\frac{\rho_2}{\rho_1}\right)} V_0 \frac{1}{\rho} e^{-jkz}$$

$$E_\phi^+ = 0$$

$$E_z^+ = 0$$

$$\boldsymbol{H}^+ = \frac{1}{\eta}\boldsymbol{a}_z \times \boldsymbol{E}^+$$

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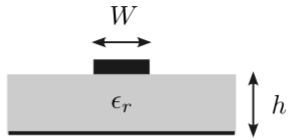
Microwave Circuits
Guiding Structures

Microstrip Tansmission Line

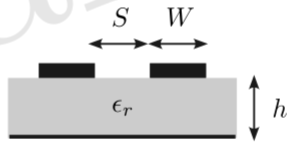
- The TEM^z mode can't propagate in a microstrip transmission line. This is due to the fact that the z component of both the electric and the magnetic fields is non-zero at the air-dielectric interface.
  At the interface, the boundary conditions for $\boldsymbol{E}$ and $\boldsymbol{H}$ require that $E_x^a = E_x^d$, $E_y^a = \epsilon_r E_y^d$, $H_x^a = H_x^d$ and $H_y^a = H_y^d$; Since $H_x = j\frac{1}{\omega\mu}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)$, it follows that $(\epsilon_r - 1)\frac{\partial E_y^d}{\partial z} = \frac{\partial E_z^a}{\partial y} - \frac{\partial E_z^d}{\partial y}$ which requires that $E_z \neq 0$. Similarly, since $E_x = -j\frac{1}{\omega\epsilon}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)$, it follows that $(\epsilon_r - 1)\frac{\partial H_y}{\partial z} = \epsilon_r \frac{\partial H_z^a}{\partial y} - \frac{\partial H_z^d}{\partial y}$ which requires that $H_z \neq 0$.
  Since the fields in the air are fringe fields and are, therefore, negligible, the **dominant mode** of propagation is **assumed to be** TEM^z (also referred to as Quasi-TEM).
- The microstrip line is modeled as a metallic strip fully imbedded in a dielectric medium with an effective dielectric constant given by $\epsilon_r^{\text{eff}} = \frac{C}{C_a}$ in which C is the capacitance of the microstrip line structure and C_a is the capacitance of the microstrip line structure with its dielectric removed (i.e., an air-filled structure). The guided wavelength is given by $\lambda_g = \frac{\lambda_0}{\sqrt{\epsilon_r^{\text{eff}}}}$.

  The above approach to obtaining the ϵ_r^{eff} is based on equating the phase constant of the wave propagating in the fully-imbedded model of the microstrip line with the phase constant of its equivalent transmission line model which results in $\epsilon_r^{\text{eff}} = \frac{1}{\mu_0\epsilon_0}LC$. Since the inductance does not depend on the presence of the dielectric, its value can be replaced by $L = \mu_0\epsilon_0\frac{1}{C_a}$ (obtained from the air-filled structure) which yields the above-mentioned result.
- For the microstrip line, the structural parameter $\frac{W}{h}$ (and also the λ_g) is obtained from the desired **characteristic impedance** Z_0 at the frequency of operation.



Coupled Microstrip Tansmission Line

- For the coupled microstrip line, the structural parameters $\frac{W}{h}$ and $\frac{S}{h}$ (and also the λ_g) are obtained from the desired **even and odd mode characteristic impedances** Z_{0e} and Z_{0o} at the frequency of operation.



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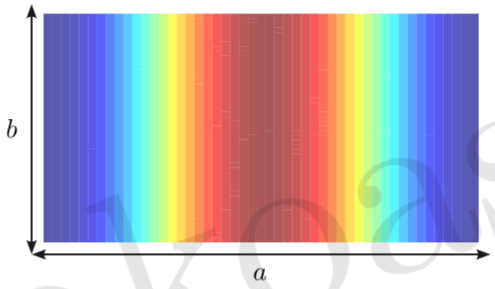
Microwave Circuits
Guiding Structures

Rectangular Waveguide

- The TEM^z mode can't propagate in waveguides. The solution of the laplace equation over the cross section is a constant (zero) function and, therefore, its gradient (which gives the electric field) is zero.
- The **dominant mode** of propagation is TE^z_{10} with its cutoff frequency given by $f_c^{\mathrm{TE}^z_{10}} = \frac{c_0}{2a\sqrt{\epsilon_r}}$. The guided wavelength and the wave impedance are given by $\lambda_g = \frac{\lambda_0}{\sqrt{1-\left(\frac{f_c^{\mathrm{TE}^z_{10}}}{f}\right)^2}}$ and $Z_{\mathrm{TE}^z_{10}} = \frac{1}{\sqrt{\epsilon_r}} \frac{\eta_0}{\sqrt{1-\left(\frac{f_c^{\mathrm{TE}^z_{10}}}{f}\right)^2}}$.
  The **next higher-order mode**, when $\frac{a}{b} = 2$, is TE^z_{01} with its cutoff frequency given by $f_c^{\mathrm{TE}^z_{01}} = 2 f_c^{\mathrm{TE}^z_{10}}$.
- From the transmission line model, for a lossless transmission line $Z_0 = \frac{2P_{\mathrm{av}}}{|I^+|^2}$ (or equivalently, $Z_0 = \frac{|V^+|^2}{2P_{\mathrm{av}}}$). Since $I^+ = \int_0^a J_z\left(x, y=0, z=0\right) dx$ in which $J_z = -H_x$, it follows that the **characteristic impedance** for the rectangular waveguide is given by $Z_0 = \frac{\pi^2}{8}\frac{b}{a} Z_{\mathrm{TE}^z_{10}}$.
- The amplitude of the TE^z_{10} mode is determined by the electric current probe that is used to excite it. Assuming a probe of height h supporting a known current density $\boldsymbol{J}(\boldsymbol{r}) = I_0 \delta\left(x - \frac{a}{2}\right) \frac{\sin(k(h-y))}{\sin(kh)} \delta\left(z\right) \boldsymbol{a}_y$ and taking the amplitude of the E_y^+ as reference, i.e., $E_y^+ = A^+ \sin\left(\frac{x}{a}\pi\right) e^{-jk_z z}$, then the quantity $K^+ \equiv \frac{A^+}{I_0}$ is given by $K^+ = -\frac{1}{a}\frac{\tan\left(\frac{kh}{2}\right)}{kb} Z_{\mathrm{TE}^z_{10}}$.



$$H^+_{x,\,\mathrm{TE}^z_{10}} = j\frac{ak\sqrt{1-\left(\frac{f_c^{10}}{f}\right)^2}}{\pi} H_0^+ \sin\left(\frac{x}{a}\pi\right) e^{-jk\sqrt{1-\left(\frac{f_c^{10}}{f}\right)^2}z}$$

$$H^+_{y,\,\mathrm{TE}^z_{10}} = 0$$

$$H^+_{z,\,\mathrm{TE}^z_{10}} = H_0^+ \cos\left(\frac{x}{a}\pi\right) e^{-jk\sqrt{1-\left(\frac{f_c^{10}}{f}\right)^2}z}$$

$$\boldsymbol{E}^+_{\mathrm{TE}^z_{10}} = -Z_{\mathrm{TE}^z_{10}} \boldsymbol{a}_z \times \boldsymbol{H}^+_{\mathrm{TE}^z_{10}}$$

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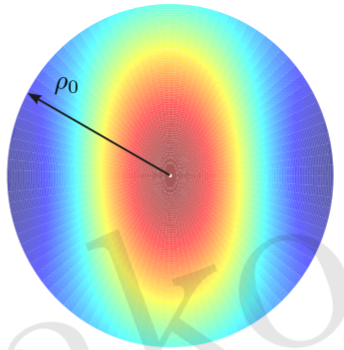
Microwave Circuits
Guiding Structures

Circular Waveguide

- The TEM^z mode can't propagate in waveguides. The solution of the Laplace equation over the cross section is a constant (zero) function and, therefore, its gradient (which gives the electric field) is zero.
- The **dominant mode** of propagation is TE_{11}^z with its cutoff frequency given by $f_c^{\text{TE}_{11}^z} = \frac{\chi_{11}^{'} c_0}{2\pi\rho_0\sqrt{\epsilon_r}}$. The guided wavelength and the wave impedance are given by $\lambda_g = \frac{\lambda_0}{\sqrt{1-\left(\frac{f_c^{\text{TE}_{11}^z}}{f}\right)^2}}$ and $Z_{\text{TE}_{11}^z} = \frac{1}{\sqrt{\epsilon_r}} \frac{\eta_0}{\sqrt{1-\left(\frac{f_c^{\text{TE}_{11}^z}}{f}\right)^2}}$.
  The **next higher-order mode** is TM_{01}^z with its cutoff frequency given by $f_c^{\text{TM}_{01}^z} = 1.31 f_c^{\text{TE}_{11}^z}$.
- From the transmission line model, for a lossless transmission line $Z_0 = \frac{2P_{\text{av}}}{|I^+|^2}$ (or equivalently, $Z_0 = \frac{|V^+|^2}{2P_{\text{av}}}$). Since $I^+ = \int_0^\pi J_z\left(\rho = \rho_0, \phi, z = 0\right)\rho_0 d\phi$ in which $J_z = -H_\phi$, it follows that the **characteristic impedance** for the circular waveguide is given by $Z_0 = \frac{\pi}{8}\left(\chi_{11}^{'\,2} - 1\right) Z_{\text{TE}_{11}^z}$.



$$H^+_{\rho,\,\text{TE}^z_{11}} = \frac{\rho_0 k\sqrt{1-\left(\frac{f_c^{11}}{f}\right)^2}}{\chi_{11}^{'}} H_0^+ J_1^{'}\left(\chi_{11}^{'}\frac{\rho}{\rho_0}\right)\cos\left(\phi\right) e^{-jk\sqrt{1-\left(\frac{f_c^{11}}{f}\right)^2}z}$$

$$H^+_{\phi,\,\text{TE}^z_{11}} = \frac{\rho_0^2 k\sqrt{1-\left(\frac{f_c^{11}}{f}\right)^2}}{\chi_{11}^{'\,2}} H_0^+ \frac{J_1\left(\chi_{11}^{'}\frac{\rho}{\rho_0}\right)}{\rho}\sin\left(\phi\right) e^{-jk\sqrt{1-\left(\frac{f_c^{11}}{f}\right)^2}z}$$

$$H^+_{z,\,\text{TE}^z_{11}} = H_0^+ J_1\left(\chi_{11}^{'}\frac{\rho}{\rho_0}\right)\cos\left(\phi\right) e^{-jk\sqrt{1-\left(\frac{f_c^{11}}{f}\right)^2}z}$$

$$\boldsymbol{E}^+_{\text{TE}^z_{11}} = -Z_{\text{TE}^z_{11}}\boldsymbol{a}_z \times \boldsymbol{H}^+_{\text{TE}^z_{11}}$$

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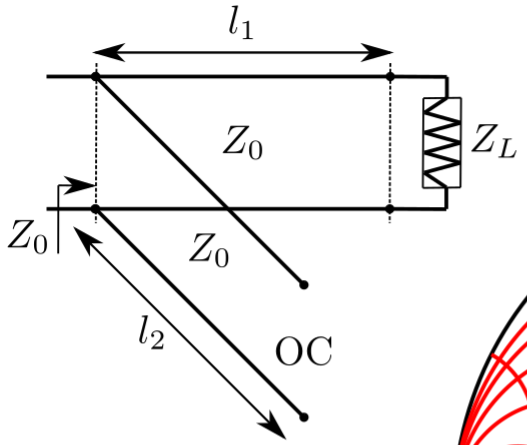
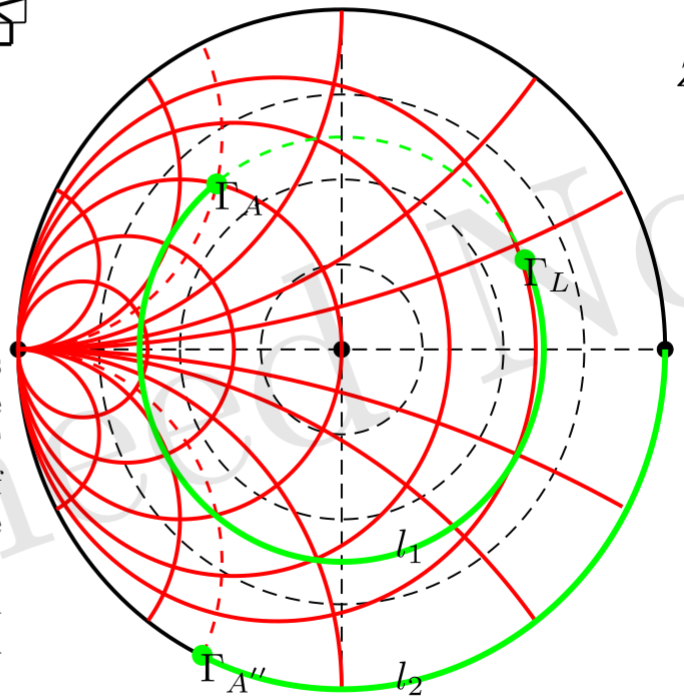
Microwave Circuits
Matching Networks

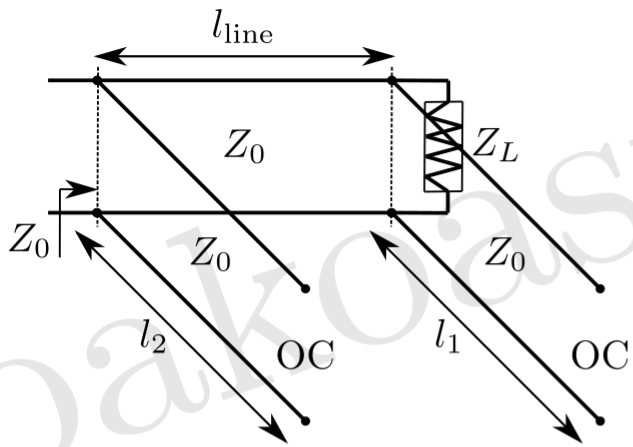
Single Stub Tuning

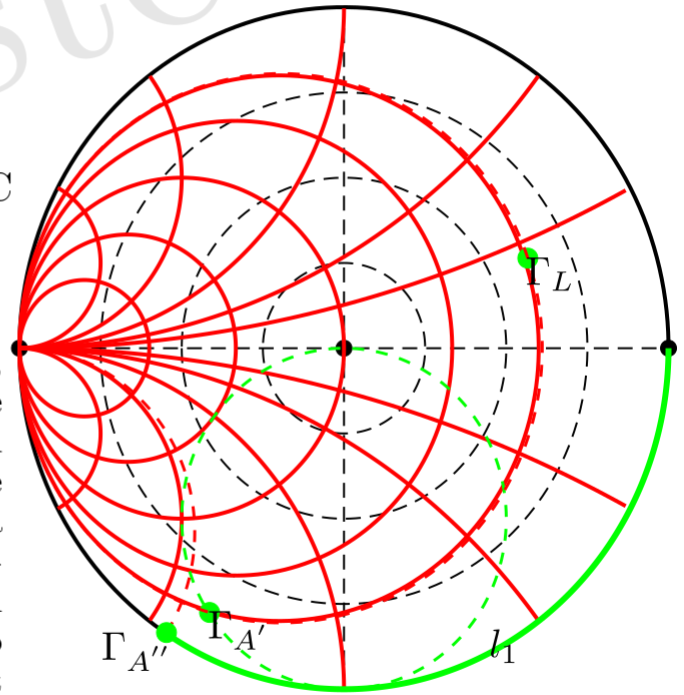




The line of length l_1 transforms Γ_L to Γ_A on the unity circle (moving it along a constant Γ circle). By connecting a stub of length l_2 (which transforms the OC to $\Gamma_{A''}$) in parallel with Y_A, its susceptance is canceled and Γ_A is transformed to the origin of the plane.

Double Stub Tuning





With a fixed line length (l_{line}), all loads that can be matched lie on a rotated unity circle (shown with dashed line for the case $l_{\text{line}} = \frac{\lambda}{8}$). By connecting a stub of length l_1 (which transforms OC to $\Gamma_{A''}$) in parallel with Y_L, Γ_L is transformed to $\Gamma_{A'}$ (moving along a constant conductance circle). The procedure continues as with the single-stub tuning.

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Microwave Circuits
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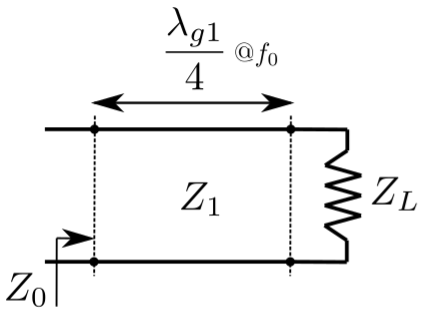
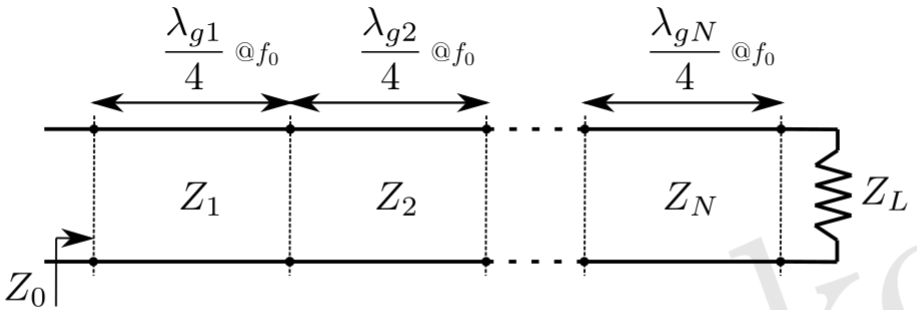
■ Quarter-Wave Transformer



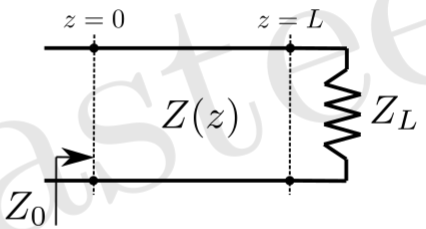
- $Z_1 = \sqrt{Z_0 Z_L}$
- $\text{BW} \equiv \dfrac{\Delta f}{f_0}$
  $= 2 - \dfrac{4}{\pi}\theta_m$
  $\theta_m = \cos^{-1}\left(\dfrac{\Gamma_m}{\sqrt{1-\Gamma_m^2}}\dfrac{2\sqrt{Z_0 Z_L}}{|Z_L - Z_0|}\right)$

■ Multi-Section Transformer



- The input reflection coefficient is given by $\Gamma(\theta) = \sum_{n=0}^{N} \Gamma_n e^{-j2n\theta}$. The Γ_n are required to be symmetric with respect to the middle element. With this configuration, the input reflection coefficient reduces to $\Gamma(\theta) = e^{-jN\theta}\left(t + 2\sum_{n=0}^{M}\Gamma_n \cos\left((N-2n)\theta\right)\right)$ in which $M = \frac{N-1}{2}$, $t = 0$ when $N \in \text{O}$ and $M = \frac{N}{2} - 1$, $t = \Gamma_{\frac{N}{2}}$ when $N \in \text{E}$.
  Once the Γ_n are determined, the segment impedances are obtained using $\frac{Z_{n+1}}{Z_n} = \frac{1+\Gamma_n}{1+\Gamma_n}$ in which $n = 0, 1, \cdots, N-1$; Furthermore, The bandwidth, defined as $\text{BW} \equiv \frac{\Delta f}{f_0}$, is given by, $\text{BW} = 2 - \frac{4}{\pi}\theta_m$.
- The Maximally-Flat Design
  The Γ_n are determined by requiring $\Gamma(\theta) = e^{-jN\theta}\left(\left[\frac{1}{2^N}\Gamma_L\right](2\cos\theta)^N\right)$. In this case, $\theta_m = \cos^{-1}\left(\left(\frac{|\Gamma_L|}{\Gamma_m}\right)^{-\frac{1}{N}}\right)$.
- The Chebyshev Design
  The Γ_n are determined by requiring $\Gamma(\theta) = e^{-jN\theta}\left(\left[\frac{\Gamma_L}{|\Gamma_L|}\Gamma_m\right]T_N(\sec\theta_m \cos\theta)\right)$ in which $\theta_m = \sec^{-1}\left(\cosh\left(\frac{1}{N}\cosh^{-1}\left(\frac{|\Gamma_L|}{\Gamma_m}\right)\right)\right)$.

■ Tapered Transformer



- The Exponential Taper Design
  $Z(z) = Z_0 e^{\ln\left(\frac{Z_L}{Z_0}\right)\left(\frac{z}{L}\right)}$, $\beta_0 L \geq \pi$
  If $f_{\min}$ is the lowest frequency of interest, then Z_L can be matched for all $f \geq f_{\min}$ by setting $L = \frac{\lambda_{g0}^{@f_{\min}}}{2}$.

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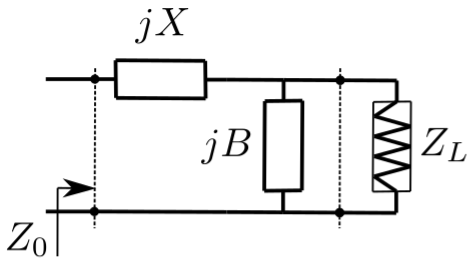
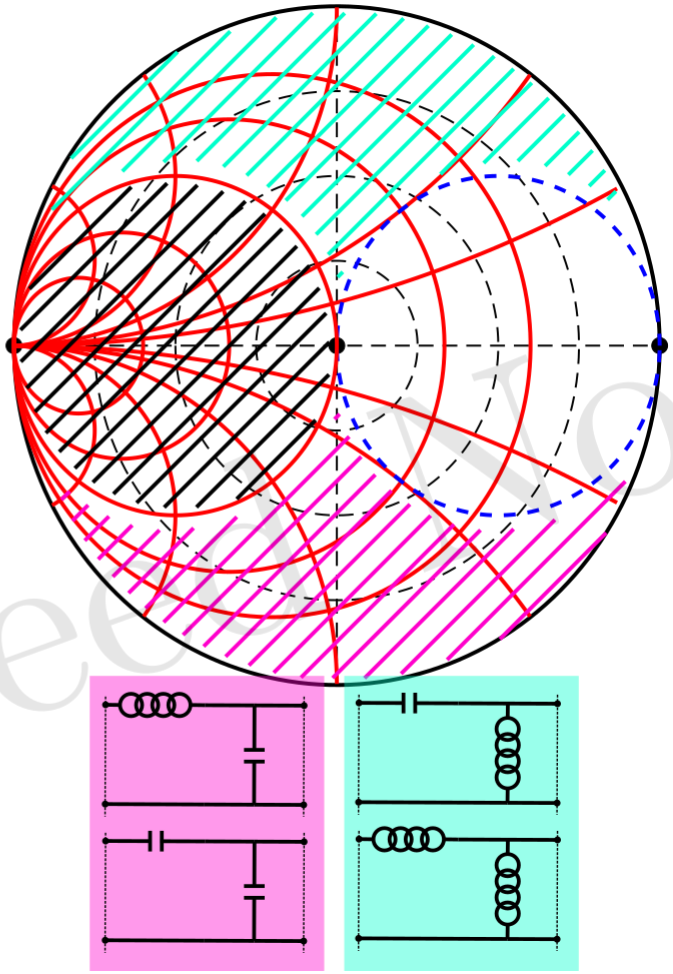
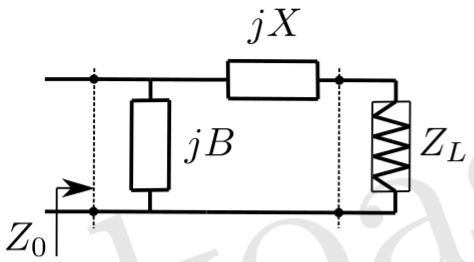
Microwave Circuits
Matching Networks

■ **Lumped-Element Matching Networks: The L Section**

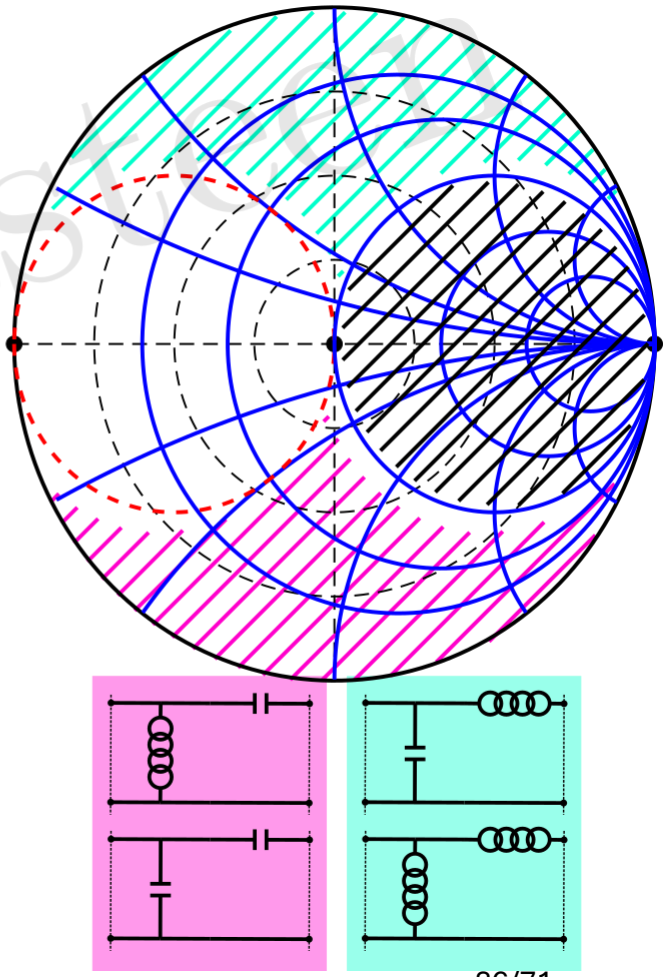


The shunt element transforms Γ_L to a point on the Z-chart's unity circle (shown with dashed line), moving it along a constant conductance circle. The series element cancels the remaining reactance and transforms the point on the unity circle to the origin of the plane.
The forbidden regions for each type of section is shown on the smith chart.





The series element transforms Γ_L to a point on the Y-chart's unity circle (shown with dashed line), moving it along a constant resistance circle. The shunt element cancels the remaining susceptance and transforms the point on the unity circle to the origin of the plane.
The forbidden regions for each type of section is shown on the smith chart.



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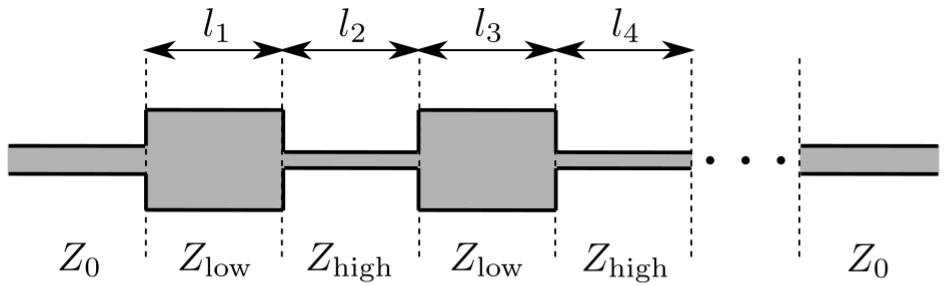
Microwave Circuits
Filter Networks

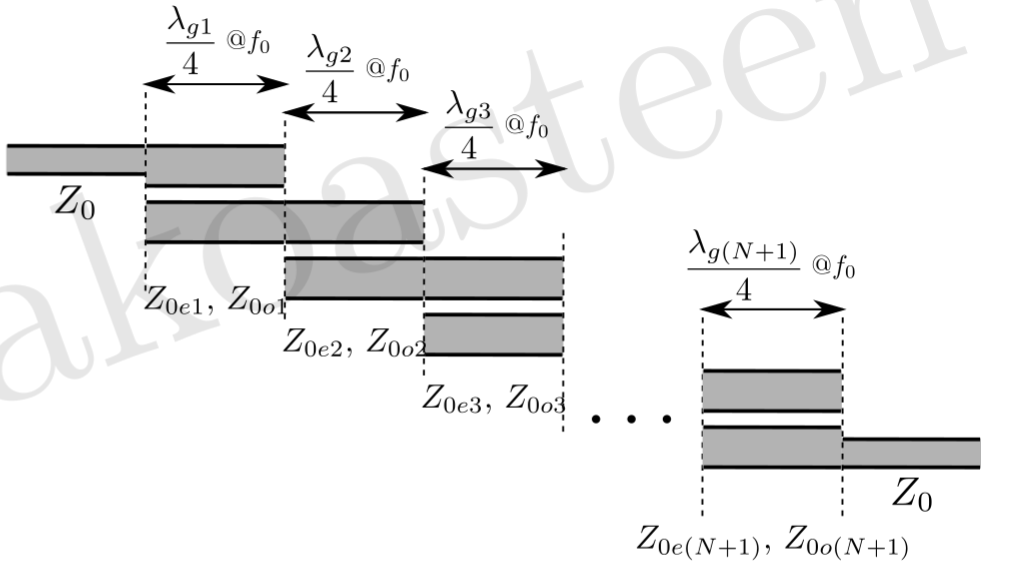
■ **Lowpass Insertion Loss:**
The Stepped-Impedance Design



- $l_{n,\,\text{low}} = \dfrac{1}{2\pi}\left(\dfrac{Z_{\text{low}}}{Z_0} g_n\right) \lambda_{g,\text{low}}^{@f_c}$

  $l_{n,\,\text{high}} = \dfrac{1}{2\pi}\left(\dfrac{Z_0}{Z_{\text{high}}} g_n\right) \lambda_{g,\text{high}}^{@f_c}$

■ **Bandpass Insertion Loss:**
The Coupled-Line (Parallel-Coupled) Design



- $Z_{0en} = Z_0\left(1 + J_n^{'} + J_n^{'2}\right)$, $Z_{0on} = Z_0\left(1 - J_n^{'} + J_n^{'2}\right)$

  In the above expressions, $J_n^{'} = \alpha_n \frac{\pi\Delta}{2} \frac{1}{\sqrt{g_n g_{n-1}}}$ for $n = 1, \cdots, N+1$ in which $\alpha_1 = \alpha_{N+1} = \sqrt{\frac{2}{\pi\Delta}}$ and $\alpha_n = 1$ for $n = 2, \cdots, N$; Δ is the fractional bandwidth, i.e., $\Delta \equiv \frac{\Delta f}{f_0}$ and $g_0 = 1$.

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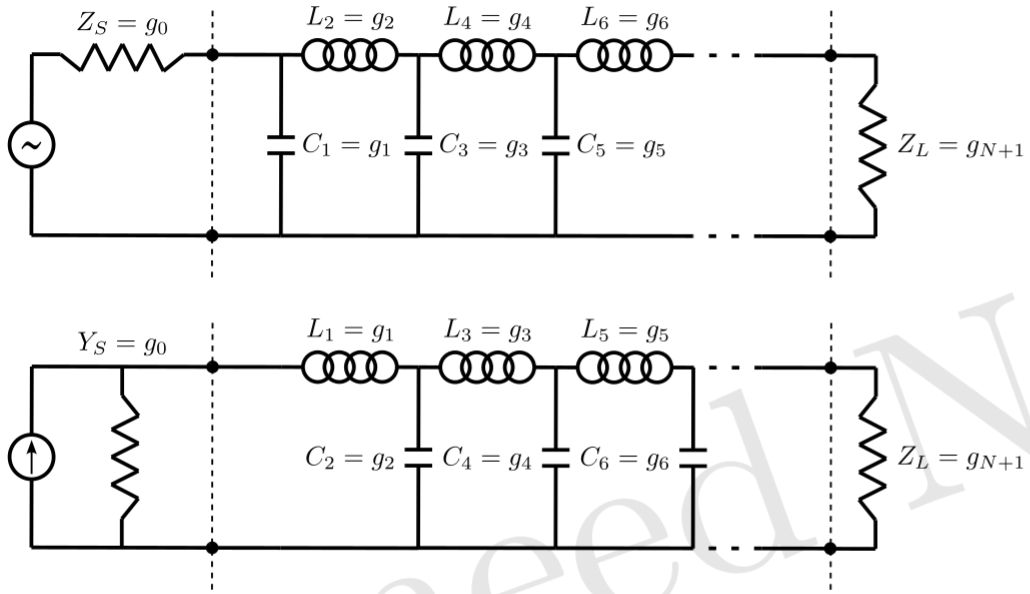
Microwave Circuits
Filter Networks

■ **Lumped-Element Filter Networks:**
The Lowpass Prototypes



■ **Lumped-Element Filter Networks:**
Lowpass Insertion Loss

In the Lowpass Prototype, the C_n are replaced with $C_n^{'} = \left(\frac{1}{\omega_c Z_0}\right) C_n$ and the L_n are replaced with $L_n^{'} = \left(\frac{Z_0}{\omega_c}\right) L_n$.

■ **Lumped-Element Filter Networks:**
Bandpass Insertion Loss

In the Lowpass Prototype, the C_n are replaced with the parallel combination of $C_n^{''}$ and $L_n^{''}$ in which $C_n^{''} = \left(\frac{1}{\omega_0 Z_0 \Delta}\right) C_n$ and $L_n^{''} = \left(\frac{Z_0 \Delta}{\omega_0}\right) \frac{1}{C_n}$. The L_n are replaced with the series combination of $C_n^{''}$ and $L_n^{''}$ in which $C_n^{''} = \left(\frac{\Delta}{\omega_0 Z_0}\right) \frac{1}{L_n}$ and $L_n^{''} = \left(\frac{Z_0}{\omega_0 \Delta}\right) L_n$. In these relations, $\omega_0 \equiv \sqrt{\omega_1 \omega_2}$ and $\Delta \equiv \frac{\Delta f}{f_0}$.

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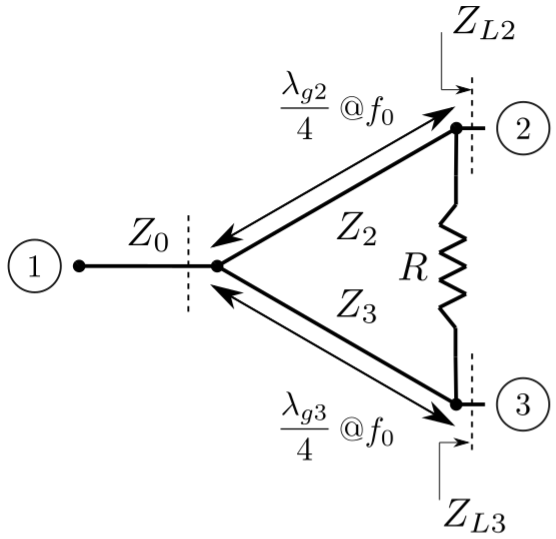
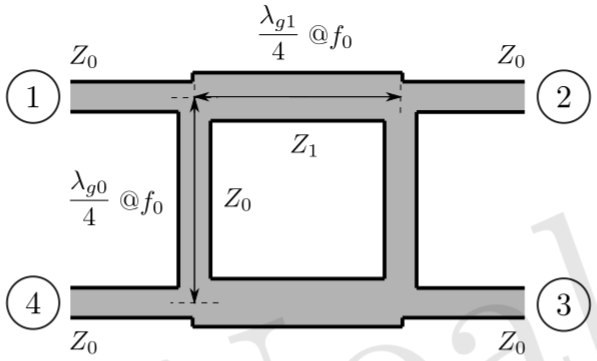
Microwave Circuits
Power Dividers / Combiners

Wilkinson Power Divider



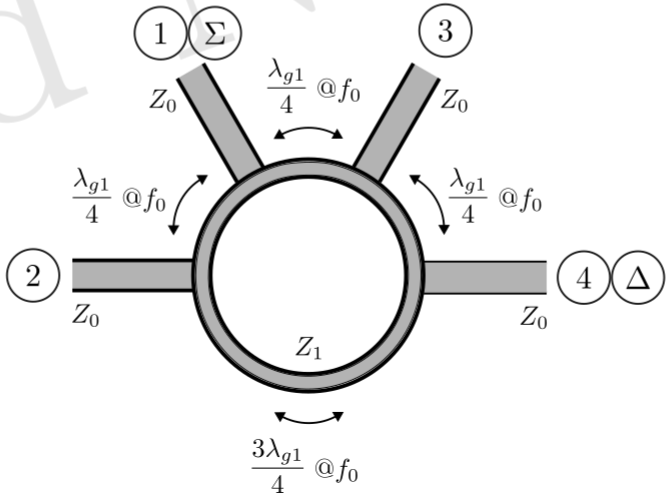
- Design Procedure for $\frac{P_3}{P_2} = K^2$:
  1. Choose Z_2
  2. $Z_{L2} = \left(\frac{1}{1+K^2}\right)\frac{Z_2^2}{Z_0}$
  3. $Z_3 = \frac{1}{K^2}Z_2,\ Z_{L3} = \frac{1}{K^2}Z_{L2}$
  4. $R = \frac{1}{K^2}\frac{Z_2^2}{Z_0}$
- The resistor R is not necessary for the normal operation of the divider. It only provides isolation between ports two and three.

3dB Coupler: The 90° Hybrid (Branch-Line Coupler)



- $Z_1 = \frac{1}{\sqrt{2}}Z_0$
- $\boldsymbol{S} = -j\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & -j & 0 \\ 1 & 0 & 0 & -j \\ -j & 0 & 0 & 1 \\ 0 & -j & 1 & 0 \end{bmatrix}$
- Port four is isolated from port one because the phase difference between the two paths between these two ports is 180°.

3dB Coupler: The 180° Hybrid (Ring Hybrid / Rat Race)



- $Z_1 = \sqrt{2}Z_0$
- $\boldsymbol{S} = -j\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$
- Port four is isolated from port one because the phase difference between the two paths between these two ports is 180°.

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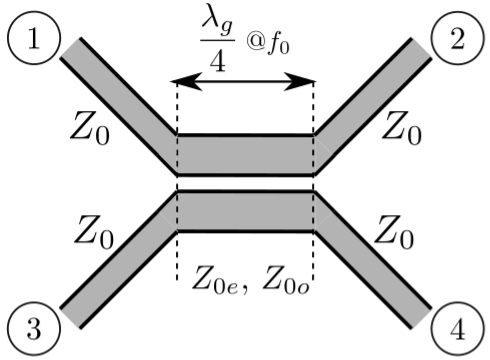
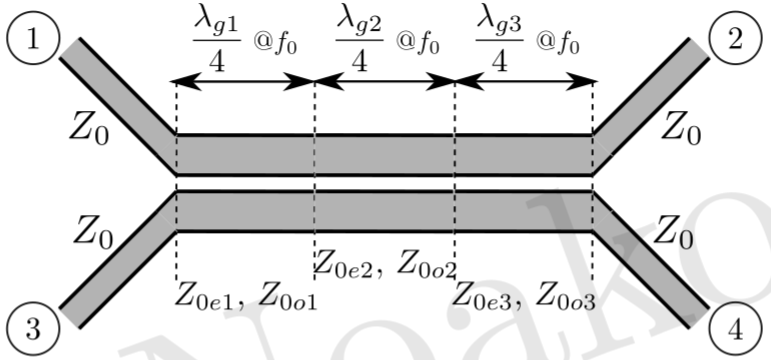
Microwave Circuits
Directional Couplers

■ Coupled-Line Coupler



- $Z_{0e} = Z_0\sqrt{\frac{1+C}{1-C}}$, $Z_{0o} = Z_0\sqrt{\frac{1-C}{1+C}}$
- $\boldsymbol{S} = -j\begin{bmatrix} 0 & \sqrt{1-C^2} & jC & 0 \\ \sqrt{1-C^2} & 0 & 0 & jC \\ jC & 0 & 0 & \sqrt{1-C^2} \\ 0 & jC & \sqrt{1-C^2} & 0 \end{bmatrix}$

■ Multi-Section Coupled-Line Coupler



- $S_{31}(\theta) = je^{-j\theta}\left(\sin\theta\sum_{n=1}^{N} C_n e^{-j2(n-1)\theta}\right)$; The C_n are required to be symmetric with respect to the middle element. With this configuration, $S_{31}(\theta) = je^{-jN\theta}\left(\sin\theta\left[t + 2\sum_{n=1}^{M} C_n\cos\left(\left(N-(2n-1)\right)\theta\right)\right]\right)$ in which $M = \frac{N-1}{2}$, $t = C_{\frac{N+1}{2}}$ when $N \in \mathrm{O}$.
- The Maximally-Flat Design

  The C_n are determined by requiring (a) $-je^{jN\theta}S_{31}(\theta) = C$ at $\theta = \frac{\pi}{2}$ and (b) $\frac{d^n}{d\theta^n}\left(-je^{jN\theta}S_{31}(\theta)\right) = 0$ for $n = 1, \cdots, N-1$ at $\theta = \frac{\pi}{2}$.

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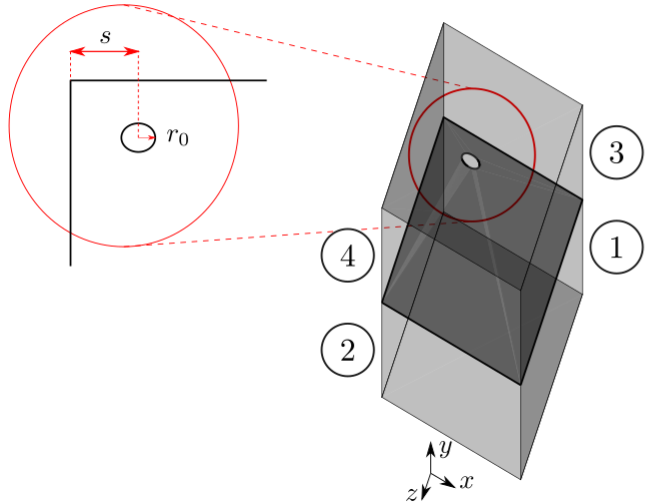
Microwave Circuits
Directional Couplers

■ Bethe Hole Coupler



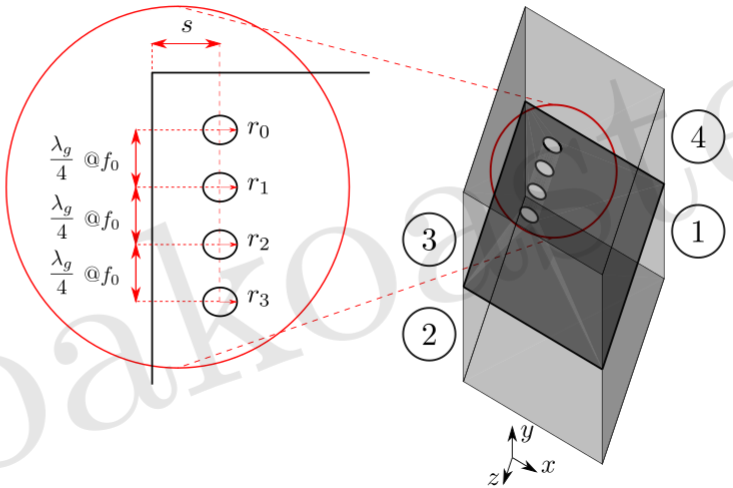
- The amplitudes of the forward and backward propagating TE_{10}^{z} mode in the top waveguide, i.e., A^+ and A^- are determined by coupling via the hole in its broad wall. Assuming a round hole of radius r, the quantities $K^+ \equiv \frac{1}{r^3}\frac{A^+}{A}$ and $K^- \equiv \frac{1}{r^3}\frac{A^-}{A}$ are given by:

$$K^+ = -j\frac{2}{3}\omega\epsilon_0\frac{Z_{\mathrm{TE}_{10}}}{ab}\left[\sin^2\left(\frac{s}{a}\pi\right) - 2\left(\frac{\eta_0}{Z_{\mathrm{TE}_{10}}}\right)^2\left(\sin^2\left(\frac{s}{a}\pi\right) + \left(\frac{\pi}{ak_z}\right)^2\cos^2\left(\frac{s}{a}\pi\right)\right)\right]$$

$$K^- = -j\frac{2}{3}\omega\epsilon_0\frac{Z_{\mathrm{TE}_{10}}}{ab}\left[\sin^2\left(\frac{s}{a}\pi\right) + 2\left(\frac{\eta_0}{Z_{\mathrm{TE}_{10}}}\right)^2\left(\sin^2\left(\frac{s}{a}\pi\right) - \left(\frac{\pi}{ak_z}\right)^2\cos^2\left(\frac{s}{a}\pi\right)\right)\right]$$

- $s = \frac{a}{\pi}\sin^{-1}\left(\frac{1}{\sqrt{2}\sqrt{1-\left(\frac{a}{\lambda_0}\right)^2}}\right)$, $r_0 = \left(\frac{1}{|K^-|C}\right)^{\frac{1}{3}}$
- $\boldsymbol{S} = \begin{bmatrix} 0 & \sqrt{1-C^2} & -jC & 0 \\ \sqrt{1-C^2} & 0 & 0 & -jC \\ -jC & 0 & 0 & \sqrt{1-C^2} \\ 0 & -jC & \sqrt{1-C^2} & 0 \end{bmatrix}$

■ Multi-Hole Coupler



- $S_{41}(\theta) = K^-\left(\sum_{n=0}^{N-1} r_n^3 e^{-j2n\theta}\right)$; The r_n are required to be symmetric with respect to the middle element. With this configuration, $S_{41}(\theta) = e^{-j(N-1)\theta}K^-\left(t + 2\sum_{n=0}^{M} r_n^3\cos\left(N-(2n+1)\right)\theta\right)$ in which $M = \frac{N-3}{2}$, $t = r^3_{\frac{N-1}{2}}$ when $N \in \mathrm{O}$ and $M = \frac{N}{2} - 1$, $t = 0$ when $N \in \mathrm{E}$.
  Once the r_n are determined, $S_{31}(\theta) = e^{-j(N-1)\theta}K^+\left(\sum_{n=0}^{N-1} r_n^3\right)$.
- The Chebyshev Design
  The r_n are determined by requiring $S_{41}(\theta) = e^{-j(N-1)\theta}K^-\left(\left[\frac{1}{|K^+|CD_{\min}}\right]T_{N-1}\left(\sec\theta_m\cos\theta\right)\right)$ in which $\theta_m = \sec^{-1}\left(\cosh\left(\frac{1}{N-1}\cosh^{-1}\left(D_{\min}\right)\right)\right)$.

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Microwave Circuits
Amplifiers and Oscillators

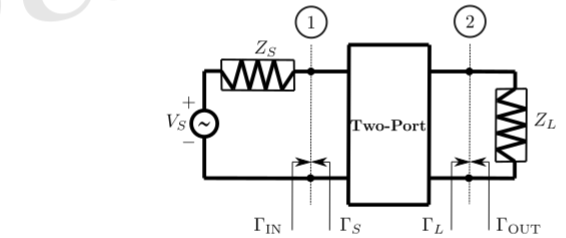
Stable Two-Ports

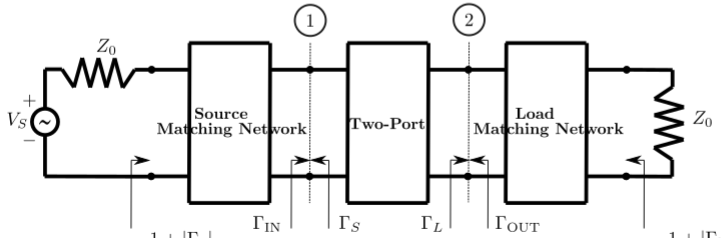
- The input and output reflection coefficients define a critical value for the source reflection coefficient and the load reflection coefficient respectively, i.e., $\Gamma_{S,c} \equiv \frac{1}{\Gamma_{\text{IN}}}$ and $\Gamma_{L,c} \equiv \frac{1}{\Gamma_{\text{OUT}}}$. These critical values are significant because, when the source or load reflection coefficients take their critical value, the positive feed-back loop of noise power at that port becomes unstable; Under this condition, which is referred to as the **barkhausen criterion**, at the input port $\Gamma_{S,c}\Gamma_{\text{IN}} = 1$ or, equivalently, $\text{Re}\,(Z_{S,c} + Z_{\text{IN}}) = 0$ and $\text{Im}\,(Z_{S,c} + Z_{\text{IN}}) = 0$. Likewise, at the output port $\Gamma_{L,c}\Gamma_{\text{OUT}} = 1$ or, equivalently, $\text{Re}\,(Z_{L,c} + Z_{\text{OUT}}) = 0$ and $\text{Im}\,(Z_{L,c} + Z_{\text{OUT}}) = 0$.
- A two-port network is stable if $\text{Re}\,(Z_S) > \text{Re}\,(Z_{S,c})$ and $\text{Re}\,(Z_L) > \text{Re}\,(Z_{L,c})$; Equivalently, $\text{Re}(Z_S + Z_{\text{IN}}) > 0$ and $\text{Re}(Z_L + Z_{\text{OUT}}) > 0$.
- In the case of a bilateral two-port network, requiring $|\Gamma_{\text{IN}}| < 1$ results in a $\Gamma_{S,c}$ that lies outside the passive region of the smith chart and, therefore, from the conditions for stability as stated above, Γ_S can take any passive value. To enforce this, Γ_L must be restricted to a region the boundary of which is determined by the output stability circle. Likewise, requiring $|\Gamma_{\text{OUT}}| < 1$ results in a $\Gamma_{L,c}$ that lies outside the passive region of the smith chart and, therefore, from the conditions for stability as stated above, Γ_L can take any passive value. To enforce this, Γ_S must be restricted to a region the boundary of which is determined by the input stability circle.
- In the case of a unilateral two-port network, $\Gamma_{\text{IN}} = S_{11}$ and $\Gamma_{\text{OUT}} = S_{22}$; Therefore, the magnitude of the input and output reflection coefficients cannot be controlled by restricting Γ_L and Γ_S respectively. If $|S_{11}| < 1$, then Γ_S can take any passive value. However, if $|S_{11}| > 1$, $\Gamma_{S,c}$ lies somewhere in the passive region of the smith chart and therefore Γ_S must be restricted to that portion of the passive region that satisfies the conditions for stability as stated above. Likewise, if $|S_{22}| < 1$, then Γ_L can take any passive value. However, if $|S_{22}| > 1$, $\Gamma_{L,c}$ lies somewhere in the passive region of the smith chart and therefore Γ_L must be restricted to that portion of the passive region that satisfies the conditions for stability as stated above.
- The figure of merit for an **unconditionally stable** two-port network is the **maximum available gain** (MAG) and for a **potentially unstable** one is the **maximum stable gain** (MSG).

Low-Noise Amplifier (LNA)

- **The Design Procedure for Bilateral Two-Ports**

1. Draw the input stability circle on the Γ_S plane and the output stability circle on the Γ_L plane. Restrict all proceeding steps to the stable regions on those planes.
2. On the Γ_S plane, draw the noise figure circle and the availabe power gain circle. Select a point that is on both circles as the source reflection coefficient and calculate the corresponding Γ_{OUT} value.
3. The initial choice for load reflection coefficient is $\Gamma_L = \Gamma_{\text{OUT}}^*$; With this choice, $\text{VSWR}_{\text{OUT}} = 1$ and VSWR_{IN} can be calculated from the relevant formulas. If VSWR_{IN} does not satisfy the design requirements, to achieve acceptable VSWR at both ports, draw a VSWR circle on the Γ_L plane and choose a point on that circle and calculate VSWR_{IN}.





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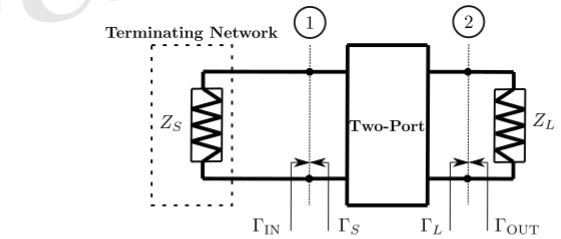
Microwave Circuits
Amplifiers and Oscillators

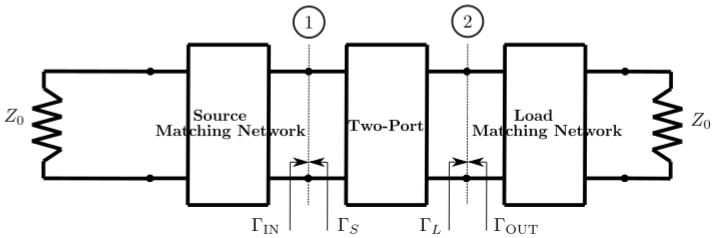
Unstable Two-Ports

- The input and output reflection coefficients define a critical value for the source reflection coefficient and the load reflection coefficient respectively, i.e., $\Gamma_{S,c} \equiv \frac{1}{\Gamma_{\text{IN}}}$ and $\Gamma_{L,c} \equiv \frac{1}{\Gamma_{\text{OUT}}}$. These critical values are significant because, when the source or load reflection coefficients take their critical value, the positive feed-back loop of noise power at that port becomes unstable; Under this condition, which is referred to as the **barkhausen criterion**, at the input port $\Gamma_{S,c}\Gamma_{\text{IN}} = 1$ or, equivalently, $\text{Re}\left(Z_{S,c} + Z_{\text{IN}}\right) = 0$ and $\text{Im}\left(Z_{S,c} + Z_{\text{IN}}\right) = 0$. Likewise, at the output port $\Gamma_{L,c}\Gamma_{\text{OUT}} = 1$ or, equivalently, $\text{Re}\left(Z_{L,c} + Z_{\text{OUT}}\right) = 0$ and $\text{Im}\left(Z_{L,c} + Z_{\text{OUT}}\right) = 0$.
- A two-port network is unstable if $\text{Re}\left(Z_S + Z_{\text{IN}}\right) \leq 0$ or $\text{Re}\left(Z_L + Z_{\text{OUT}}\right) \leq 0$. If one port satisfies the barkhausen criterion, the other one will satisfy it as well; Therefore, to induce oscillations, ensuring one of the two conditions will suffice.
- Any port's reflection coefficient is a function of the amplitude of that port's current and also the frequency, e.g., at the output port $\Gamma_{\text{OUT}} = \Gamma_{\text{OUT}}\left(A, \omega\right)$.
  Initially, $\text{Re}\left(Z_L + Z_{\text{OUT}}\left(0, \omega_0\right)\right) < 0$ must be satisfied for oscillations to start from zero amplitude. Once started, $\text{Re}\left(Z_L + Z_{\text{OUT}}\left(A_0, \omega_0\right)\right) = 0$ and $\text{Im}\left(Z_L + Z_{\text{OUT}}\left(A_0, \omega_0\right)\right) = 0$ must be satisfied for oscillations to reach a stable amplitude.
  Once oscillations have reached a stable amplitude, $\text{Re}\left(Z_{\text{OUT}}\left(A_0, \omega_0\right)\right) = \frac{1}{3}\text{Re}\left(Z_{\text{OUT}}\left(0, \omega_0\right)\right)$ must be satisfied for maximum power transfer to the load.

Negative-Resistance Oscillator

- **The Design Procedure for Bilateral Two-Ports**
  1. Draw the input stability circle on the Γ_S plane.
  2. Select a Γ_S value which is well inside the unstable region and calculate the corresponding Γ_{OUT} value.
  3. $\text{Re}\left(Z_L\right) = -\frac{1}{3}\text{Re}\left(Z_{\text{OUT}}\right)$, $\text{Im}\left(Z_L\right) = -\text{Im}\left(Z_{\text{OUT}}\right)$
- The **oscillator phase noise** (a loss quantity), denoted by $\text{L}\left(f_m\right)$, is defined as the ratio of the single-sideband (SSB) power spectral density (PSD), at a particular offset, i.e., f_m, to the the carrier power.





Oameed Noakoasteen

Microwave Circuits
Amplifiers and Oscillators

Circles: Stability, Gain, VSWR and Noise Figure

Stability and Gain Circles: Bilateral Two-Ports

Input Stability Circle

The locus of Γ_S values for which $|\Gamma_{\text{OUT}}| = 1$, is a circle.

$$c = \frac{(S_{11} - \Delta S_{22}^*)^*}{|S_{11}|^2 - |\Delta|^2},\ r = \left|\frac{S_{12}S_{21}}{|S_{11}|^2 - |\Delta|^2}\right|$$

Output Stability Circle

The locus of Γ_L values for which $|\Gamma_{\text{IN}}| = 1$, is a circle.

$$c = \frac{(S_{22} - \Delta S_{11}^*)^*}{|S_{22}|^2 - |\Delta|^2},\ r = \left|\frac{S_{12}S_{21}}{|S_{22}|^2 - |\Delta|^2}\right|$$

Available Power Gain Circle

The locus of Γ_S values for which $g_A \equiv \frac{G_A}{|S_{21}|^2}$ is constant, is a circle.

$$c = \frac{g_A (S_{11} - \Delta S_{22}^*)^*}{1 + g_A\left(|S_{11}|^2 - |\Delta|^2\right)},\ r = \frac{\left[1 - 2K|S_{12}S_{21}|\, g_A + |S_{12}S_{21}|^2 g_A^2\right]^{\frac{1}{2}}}{\left|1 + g_A\left(|S_{11}|^2 - |\Delta|^2\right)\right|}$$

Operating Power Gain Circle

The locus of Γ_L values for which $g_P \equiv \frac{G_P}{|S_{21}|^2}$ is constant, is a circle.

$$c = \frac{g_P (S_{22} - \Delta S_{11}^*)^*}{1 + g_P\left(|S_{22}|^2 - |\Delta|^2\right)},\ r = \frac{\left[1 - 2K|S_{12}S_{21}|\, g_P + |S_{12}S_{21}|^2 g_P^2\right]^{\frac{1}{2}}}{\left|1 + g_P\left(|S_{22}|^2 - |\Delta|^2\right)\right|}$$

The K

$$K \equiv \frac{1 - |S_{11}|^2 - |S_{22}|^2 + |\Delta|^2}{2|S_{12}S_{21}|}$$

Stability and Gain Circles: Unilateral Two-Ports

A two-port network is unilateral if $S_{12} = 0$. In this case, $\Gamma_{\text{IN}} = S_{11}$ and $\Gamma_{\text{OUT}} = S_{22}$. Therefore, $G_T = G_S G_0 G_L$ in which $G_S \equiv \frac{1-|\Gamma_S|^2}{|1-S_{11}\Gamma_S|^2}$, $G_0 \equiv |S_{21}|^2$ and $G_L \equiv \frac{1-|\Gamma_L|^2}{|1-S_{22}\Gamma_L|^2}$.

Input Stability Circle

The locus of Γ_S values for which $\text{Re}(Z_S + Z_{\text{IN}}) = 0$, is the circle defined by $\text{Re}(Z_S) = \text{Re}(Z_{S,c})$ in which $\Gamma_{S,c} \equiv \frac{1}{\Gamma_{\text{IN}}}$.

Output Stability Circle

The locus of Γ_L values for which $\text{Re}(Z_L + Z_{\text{OUT}}) = 0$, is the circle defined by $\text{Re}(Z_L) = \text{Re}(Z_{L,c})$ in which $\Gamma_{L,c} \equiv \frac{1}{\Gamma_{\text{OUT}}}$.

Input Gain Circle

The locus of Γ_S values for which $g_S \equiv \left(1 - |S_{11}|^2\right) G_S$ is constant, is a circle.

$$c = \frac{g_S S_{11}^*}{1 - |S_{11}|^2 (1 - g_S)},\ r = \frac{\sqrt{1 - g_S}\left(1 - |S_{11}|^2\right)}{1 - |S_{11}|^2 (1 - g_S)}$$

Output Gain Circle

The locus of Γ_L values for which $g_L \equiv \left(1 - |S_{22}|^2\right) G_L$ is constant, is a circle.

$$c = \frac{g_L S_{22}^*}{1 - |S_{22}|^2 (1 - g_L)},\ r = \frac{\sqrt{1 - g_L}\left(1 - |S_{22}|^2\right)}{1 - |S_{22}|^2 (1 - g_L)}$$

Oameed Noekoasteen

Microwave Circuits
Amplifiers and Oscillators

Circles: Stability, Gain, VSWR and Noise Figure (Cont'd)

■ **Noise Figure Circle**

The locus of Γ_S values for which $N \equiv \frac{F-F_{\min}}{4\left(\frac{R_n}{Z_0}\right)}\left|1+\Gamma_{\text{opt}}\right|^2$ is constant, is a circle.

$$c = \frac{\Gamma_{\text{opt}}}{1+N},\ r = \frac{1}{1+N}\sqrt{N^2 + N\left(1-\left|\Gamma_{\text{opt}}\right|^2\right)}$$

■ **Input VSWR Circle**

The locus of Γ_S values for which $|\Gamma_a| \equiv \frac{\text{VSWR}_{\text{IN}}-1}{\text{VSWR}_{\text{IN}}+1}$ is constant, is a circle.

$$c = \frac{\Gamma_{\text{IN}}^*\left(1-|\Gamma_a|^2\right)}{1-\left|\Gamma_a\Gamma_{\text{IN}}\right|^2},\ r = \frac{|\Gamma_a|\left(1-|\Gamma_{\text{IN}}|^2\right)}{1-\left|\Gamma_a\Gamma_{\text{IN}}\right|^2}$$

■ **Output VSWR Circle**

The locus of Γ_L values for which $|\Gamma_b| \equiv \frac{\text{VSWR}_{\text{OUT}}-1}{\text{VSWR}_{\text{OUT}}+1}$ is constant, is a circle.

$$c = \frac{\Gamma_{\text{OUT}}^*\left(1-|\Gamma_b|^2\right)}{1-\left|\Gamma_b\Gamma_{\text{OUT}}\right|^2},\ r = \frac{|\Gamma_b|\left(1-|\Gamma_{\text{OUT}}|^2\right)}{1-\left|\Gamma_b\Gamma_{\text{OUT}}\right|^2}$$

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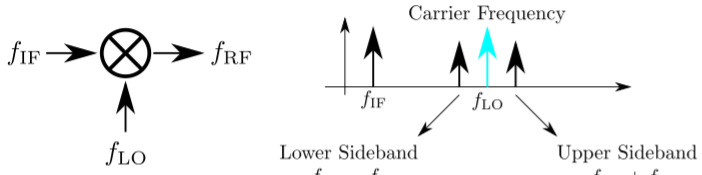
Microwave Circuits
Mixers

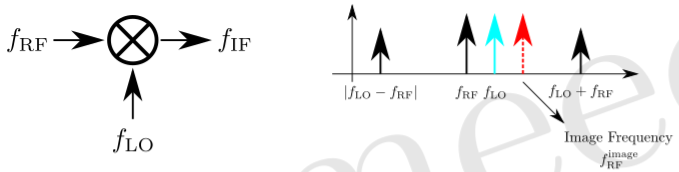
Frequency Conversion

Up Conversion



$$
\begin{aligned}
v_{\mathrm{LO}}(t) &= V_{\mathrm{LO}} \cos(\omega_{\mathrm{LO}} t) \\
v_{\mathrm{IF}}(t) &= V_{\mathrm{IF}} \cos(\omega_{\mathrm{IF}} t) \\
v_{\mathrm{MIX}}(t) &\equiv G v_{\mathrm{IF}}(t) v_{\mathrm{LO}}(t) \\
&= \frac{G V_{\mathrm{IF}} V_{\mathrm{LO}}}{2} \left[\cos\left((\omega_{\mathrm{LO}} + \omega_{\mathrm{IF}})t\right) + \cos\left((\omega_{\mathrm{LO}} - \omega_{\mathrm{IF}})t\right)\right] \\
v_{RF}(t) &= v_{\mathrm{MIX}}(t)
\end{aligned}
$$

Down Conversion



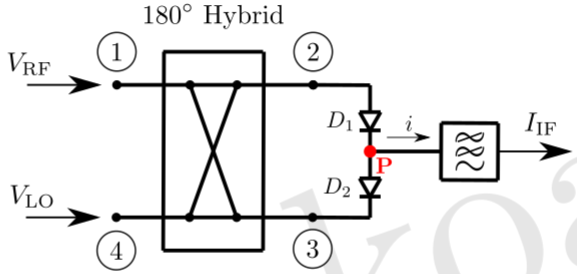
$$
\begin{aligned}
v_{\mathrm{LO}}(t) &= V_{\mathrm{LO}} \cos(\omega_{\mathrm{LO}} t) \\
v_{\mathrm{RF}}(t) &= V_{\mathrm{RF}} \cos(\omega_{\mathrm{RF}} t) \\
v_{\mathrm{MIX}}(t) &\equiv G v_{\mathrm{RF}}(t) v_{\mathrm{LO}}(t) \\
&= \frac{G V_{\mathrm{RF}} V_{\mathrm{LO}}}{2} \left[\cos\left((\omega_{\mathrm{LO}} + \omega_{\mathrm{RF}})t\right) + \cos\left(|\omega_{\mathrm{LO}} - \omega_{\mathrm{RF}}|\, t\right)\right] \\
v_{\mathrm{IF}}(t) &= \frac{G V_{\mathrm{RF}} V_{\mathrm{LO}}}{2} \cos\left(|\omega_{\mathrm{LO}} - \omega_{\mathrm{RF}}|\, t\right)
\end{aligned}
$$

Conversion Loss

$$
L_c^{\mathrm{dB}} \equiv 10 \log \left(\frac{\text{Available Input Power}}{\text{Available Output Power}} \right)
$$

Down Converter Circuits

Single-Balanced Diode Mixer



The voltage at ports two and three are given by $v_2 = v_{\mathrm{RF}} - v_{\mathrm{LO}}$ and $v_3 = v_{\mathrm{RF}} + v_{\mathrm{LO}}$. Since diodes D_1 and D_2 are identical, it follows that point P is the ground for v_{LO}.

In the positive half-cycle of the oscillator, both diodes are off and, therefore, the voltage at point P is v_{RF}. In the negative half-cycle, both diodes are on and, also, the condition of being the ground for v_{RF} is forced on point P. As such, v_P is v_{RF} in one half-cycle and is zero in the next. Therefore, v_P can be expressed as v_{RF} multiplied by a periodic pulse with the same period as v_{LO}; It is this multiplication that produces the IF component of the voltage.

In the positive half-cycle of the oscillator, both diodes are off and, therefore, the current i is zero. In the negative half-cycle, both diodes are on and, also, the condition of being the ground for v_{RF} is forced on point P. Therefore, v_2 and $-v_3$ drop across D_1 and D_2 respectively and since $i = i_{D_1} - i_{D_2}$, the current i contains the IF component produced by both diodes. Furthermore, multiplying i by a periodic pulse (with the same period as v_{LO}) to account for its off-on switching, won't affect the current's IF component due to the existence of a DC term in the pulse's Fourier series representation.

Omeed Nakoasteen

Computational EM
An Overview of Full-Wave (Low Freq.) Methods

Partial Differential Equations (PDEs)

Frequency Domain

Finite Difference Method (FDM)

- **Electrostatic Potential**

  Laplace Equation

  $$\nabla^2 V = 0$$

Finite Element Method (FEM)

- **TM and TE Modes of Propagation and Resonance**

  Vector Potential Wave Equation

  $$\boldsymbol{\nabla}^2 \boldsymbol{A} + k^2 \boldsymbol{A} = 0$$

  $$\boldsymbol{\nabla}^2 \boldsymbol{F} + k^2 \boldsymbol{F} = 0$$

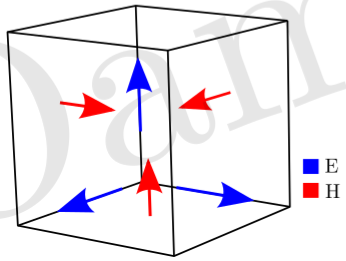
- **Electrostatic Potential**

  Laplace Equation

  $$\nabla^2 V = 0$$

Time Domain

Finite Difference Time-Domain (FDTD)



The FDTD is a time-marching scheme for electric and magnetic fields in a region of space. The first-order, time-domain PDEs of Maxwell's system (specifically, the two curl equations) are discretized in spatial dimensions using the Yee grid and are forwarded in time using the interleaved leapfrog procedure.

Integral Equations (IEs)

Method of Moments (MoM)

- **Scattering from PEC Objects**

  Electric Field Integral Equation (EFIE)

  $$\boldsymbol{n} \times \left[-j\frac{\eta}{k} \left(k^2 \iint_{S'} \boldsymbol{J}_s(\boldsymbol{r}_b') G(\boldsymbol{r}_b, \boldsymbol{r}_b')\, dS' + \boldsymbol{\nabla} \iint_{S'} \boldsymbol{\nabla}' \cdot \boldsymbol{J}_s(\boldsymbol{r}_b') G(\boldsymbol{r}_b, \boldsymbol{r}_b') \right) \right] = -\boldsymbol{n} \times \boldsymbol{E}_{inc}(\boldsymbol{r} = \boldsymbol{r}_b)$$

  Magnetic Field Integral Equation (MFIE)

  $$\boldsymbol{n} \times \left[\iint_{S'} \boldsymbol{J}_s(\boldsymbol{r}_b') \times \boldsymbol{\nabla}' G(\boldsymbol{r}_b, \boldsymbol{r}_b')\, dS' \right] + \boldsymbol{J}_s(\boldsymbol{r}_b) = \boldsymbol{n} \times \boldsymbol{H}_{inc}(\boldsymbol{r} = \boldsymbol{r}_b)$$

- **Electrostatic Charge Distribution**

  General Solution of Poisson's Equation

  $$V(\boldsymbol{r}) = \frac{1}{\epsilon} \iiint_{v'} \rho(\boldsymbol{r}') \frac{1}{4\pi \left| \boldsymbol{r} - \boldsymbol{r}' \right|}\, dv'$$

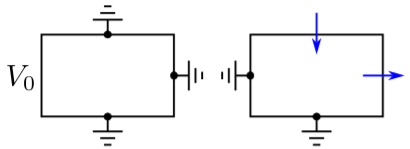
Omeed Noakoasteen

Computational EM
Some Toy Problems for Freq. Domain Methods

Finite Difference Method (FDM)

2D Electrostatic Potential Distribution

- $\nabla^2_{x,y} V = 0$

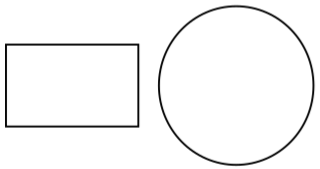


Finite Element Method (FEM)

2D Cavity Resonator

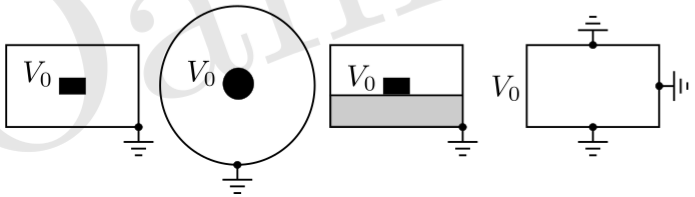
- TM^z Mode

$$\nabla^2_{x,y} A_z + k^2 A_z = 0$$



2D Electrostatic Potential Distribution

- $\nabla^2_{x,y} V = 0$

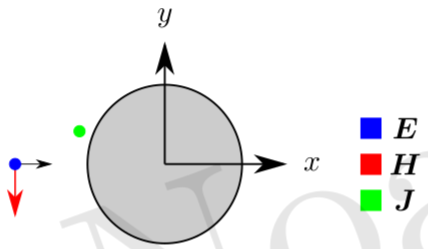


Method of Moments (MoM)

2D Scattering from Circular Cylinder

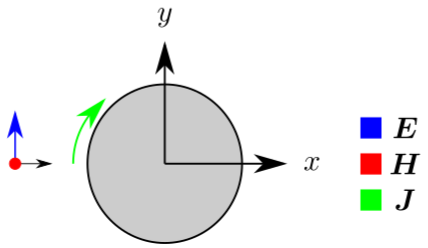
- **EFIE for TM^z Polarized Wave**

$$\frac{\eta k}{4} \int_C J_s\left(\boldsymbol{\rho}_b'\right) H_0^{(2)}\left(k\left|\boldsymbol{\rho}_b - \boldsymbol{\rho}_b'\right|\right) dC' = E_{inc}\left(\boldsymbol{\rho} = \boldsymbol{\rho}_b\right)$$



- **MFIE for TE^z Polarized Wave**

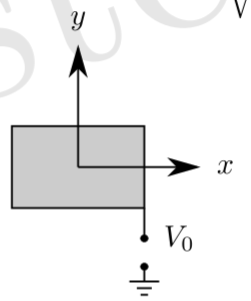
$$J_s(\boldsymbol{\rho}_b) - j\frac{1}{4} \int_C J_s\left(\boldsymbol{\rho}_b'\right) \left(\boldsymbol{\nabla}' H_0^{(2)}\left(k\left|\boldsymbol{\rho} - \boldsymbol{\rho}'\right|\right) \cdot \boldsymbol{n}'\right) dC' = -H_{inc}\left(\boldsymbol{\rho} = \boldsymbol{\rho}_b\right)$$



2D Electrostatic Charge Distribution

- **Conducting Plate Held at Constant Potential**

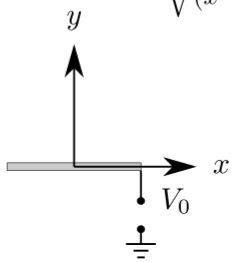
$$\frac{1}{4\pi\epsilon} \iint \rho\left(x', y'\right) \frac{1}{\sqrt{(x-x')^2 + (y-y')^2}} dx' dy' = V_0$$



1D Electrostatic Charge Distribution

- **Conducting Rod Held at Constant Potential**

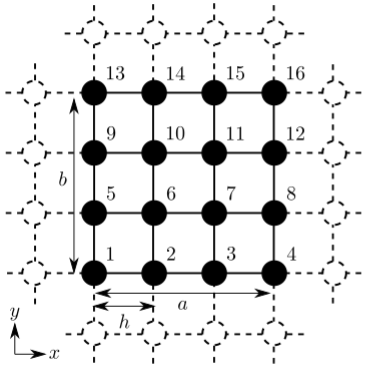
$$\frac{1}{4\pi\epsilon} \int \rho\left(x'\right) \frac{1}{\sqrt{(x-x')^2 + h^2}} dx' = V_0$$



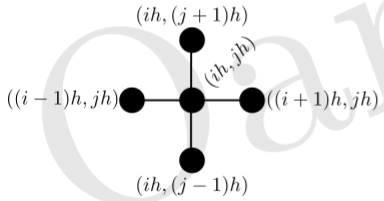
Oarneed Noakoastcen

Computational EM
Finite Difference Method (FDM)

$$\nabla^2 V(x,y) = 0$$

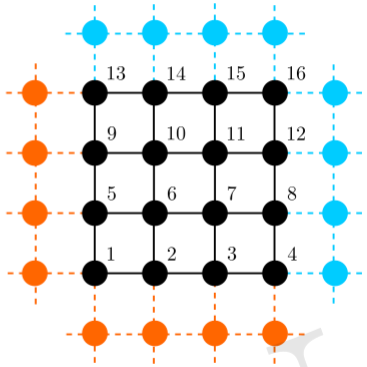


$$\nabla^2 V(ih,jh) \approx -\frac{4}{h^2}V(ih,jh) + \frac{1}{h^2}V\left((i+1)h,jh\right) + \frac{1}{h^2}V\left((i-1)h,jh\right) + \frac{1}{h^2}V\left(ih,(j+1)h\right) + \frac{1}{h^2}V\left(ih,(j-1)h\right) = 0$$



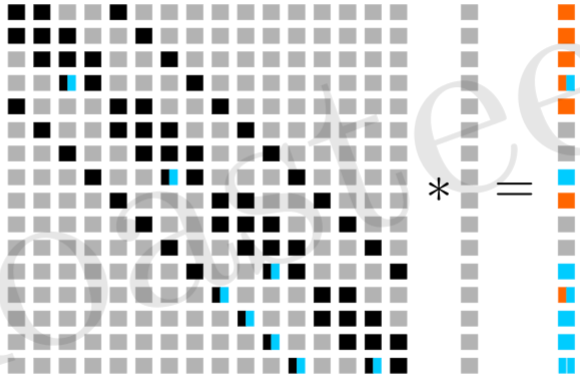
$$\frac{\partial V(x,b)}{\partial y} = E_1$$

$$V(0,y) = V_2$$



$$\frac{\partial V(a,y)}{\partial x} = E_0$$

$$V(x,0) = V_3$$



#1 : Corner (0D) Boundary Node with Dirichlet-Dirichlet BC

$$\frac{1}{h^2}\left(-4v_1 + v_2 + v_5\right) = -\frac{1}{h^2}\left(V_2 + V_3\right)$$

#5 : Edge (1D) Boundary Node with Dirichlet BC

$$\frac{1}{h^2}\left(v_1 - 4v_5 + v_6 + v_9\right) = -\frac{1}{h^2}\left(V_2\right)$$

#6 : Interior Node

$$\frac{1}{h^2}\left(v_2 + v_5 - 4v_6 + v_7 + v_{10}\right) = 0$$

#8 : Edge (1D) Boundary Node with Neumann BC

$$\frac{1}{h^2}\left(v_4 + v_7 - 4v_8 + v_{12} + v^N_{b,r,8}\right) = 0,\ v^N_{b,r,8} = v_7 + 2hE_0$$

#13: Corner (0D) Boundary Node with Dirichlet-Neumann BC

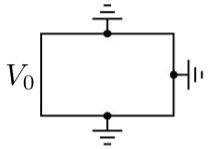
$$\frac{1}{h^2}\left(v_9 - 4v_{13} + v_{14} + v^N_{b,u,13}\right) = -\frac{1}{h^2}\left(V_2\right),\ v^N_{b,u,13} = v_9 + 2hE_1$$

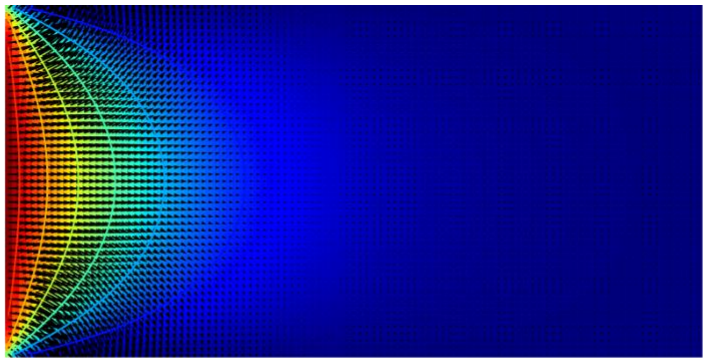
#16: Corner (0D) Boundary Node with Neumann-Neumann BC

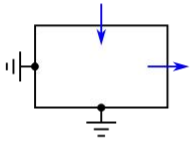
$$\frac{1}{h^2}\left(v_{12} + v_{15} - 4v_{16} + v^N_{b,u,16} + v^N_{b,r,16}\right) = 0,\ v^N_{b,u,16} = v_{12} + 2hE_1,\ v^N_{b,r,16} = v_{15} + 2hE_0$$

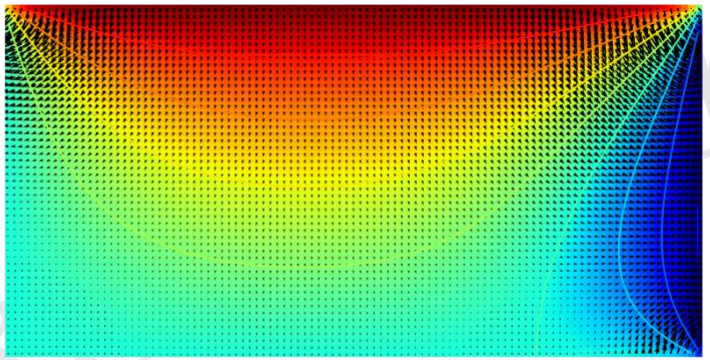
Qameeq Noakhaileen

Computational EM
Finite Difference Method (FDM)



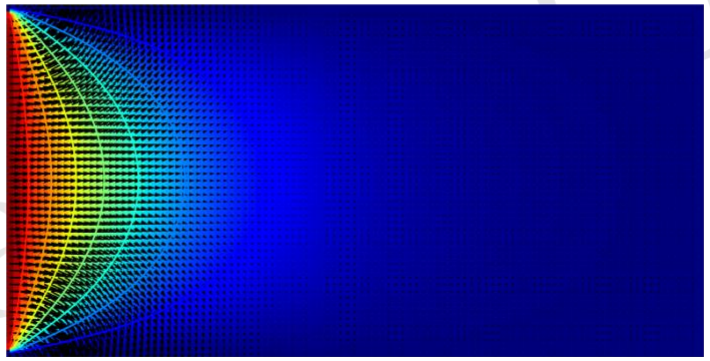






Analytic Solution

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{n\left(\sinh n\frac{\pi}{b}a\right)} \sinh n\frac{\pi}{b}(a-x) \sin n\frac{\pi}{b}y, \;\; n \in \text{Odd}$$



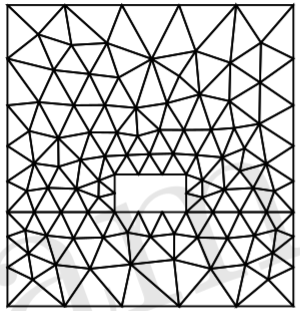
Oa Noak

Computational EM
Finite Element Method (FEM)

■ $\frac{\partial}{\partial x}\left(\alpha\frac{\partial\psi(x,y)}{\partial x}\right)+\frac{\partial}{\partial y}\left(\alpha\frac{\partial\psi(x,y)}{\partial y}\right)+\beta\psi(x,y)=f(x,y)$

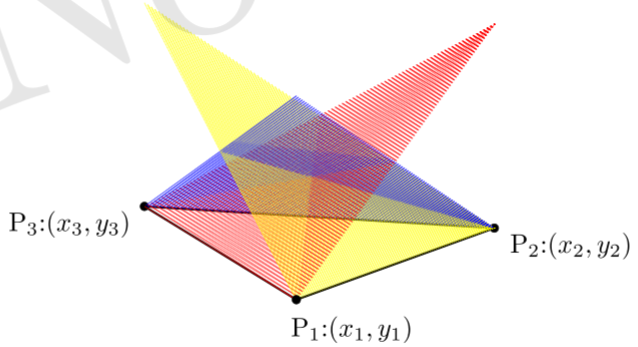
	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\boldsymbol{f}$
2D TEz	$\frac{1}{\epsilon_r}$	$k_0^2\mu_r$	$-\mu_0 M_z$
2D TMz	$\frac{1}{\mu_r}$	$k_0^2\epsilon_r$	$-\mu_0 J_z$
2D Electrostatics	ϵ_r	0	$-\frac{1}{\epsilon_0}\rho_e$
2D Magnetostatics	$\frac{1}{\mu_r}$	0	$-\mu_0 J_z$

■ A given geometric domain Ω is divided into subdomains called **Finite Elements** Ω_e. The collection of finite elements is called **Finite Element Mesh** which represents a discretization of Ω. For 2D domains, the most common element shape is **Triangular**.



Linear Element

■ Triangular Ω_e is defined by its three points which are numbered counterclockwise. Over Ω_e, the unknown function $\psi(x,y)$ is represented using a set of three **Linear Approximation Functions** $\phi_i(x,y)$ such that at any point in Ω_e, $\Psi(x,y)\approx\sum_{i=1}^{3}\Psi_i\phi_i(x,y)$. In this formulation, Ψ_i refere to the unknown function evaluated at the triangle's points, i.e., $\Psi_i\equiv\Psi(\mathrm{P}_i)$.
Linear approximation functions are defined such that $\phi_i(\mathrm{P}_j)=1$ if $j=i$ and $\phi_i(\mathrm{P}_j)=0$ if $j\neq i$; and that $\sum_{i=1}^{3}\phi_i(x,y)=1$.



Element Model

■ $\boldsymbol{K}_e\boldsymbol{\psi}_e=\boldsymbol{Q}_e+\boldsymbol{f}_e,\ \boldsymbol{K}_e\equiv\boldsymbol{M}_e+\boldsymbol{T}_e$

$$\boldsymbol{M}_e=-\frac{\alpha}{4S}\begin{bmatrix}M_{11}&M_{12}&M_{13}\\M_{12}&M_{22}&M_{23}\\M_{13}&M_{23}&M_{33}\end{bmatrix}$$

$$\boldsymbol{T}_e=\frac{\beta S}{12}\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}$$

$$\boldsymbol{f}_e=\frac{f_e S}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$M_{11}=(y_2-y_3)^2+(x_3-x_2)^2$$
$$M_{12}=(y_2-y_3)(y_3-y_1)+(x_3-x_2)(x_1-x_3)$$
$$M_{13}=(y_2-y_3)(y_1-y_2)+(x_3-x_2)(x_2-x_1)$$
$$M_{22}=(y_3-y_1)^2+(x_1-x_3)^2$$
$$M_{23}=(y_3-y_1)(y_1-y_2)+(x_1-x_3)(x_2-x_1)$$
$$M_{33}=(y_1-y_2)^2+(x_2-x_1)^2$$

$$S\equiv\frac{1}{2}\det(\boldsymbol{J}),\ \boldsymbol{J}\equiv\begin{bmatrix}x_2-x_1&y_2-y_1\\x_3-x_1&y_3-y_1\end{bmatrix}$$

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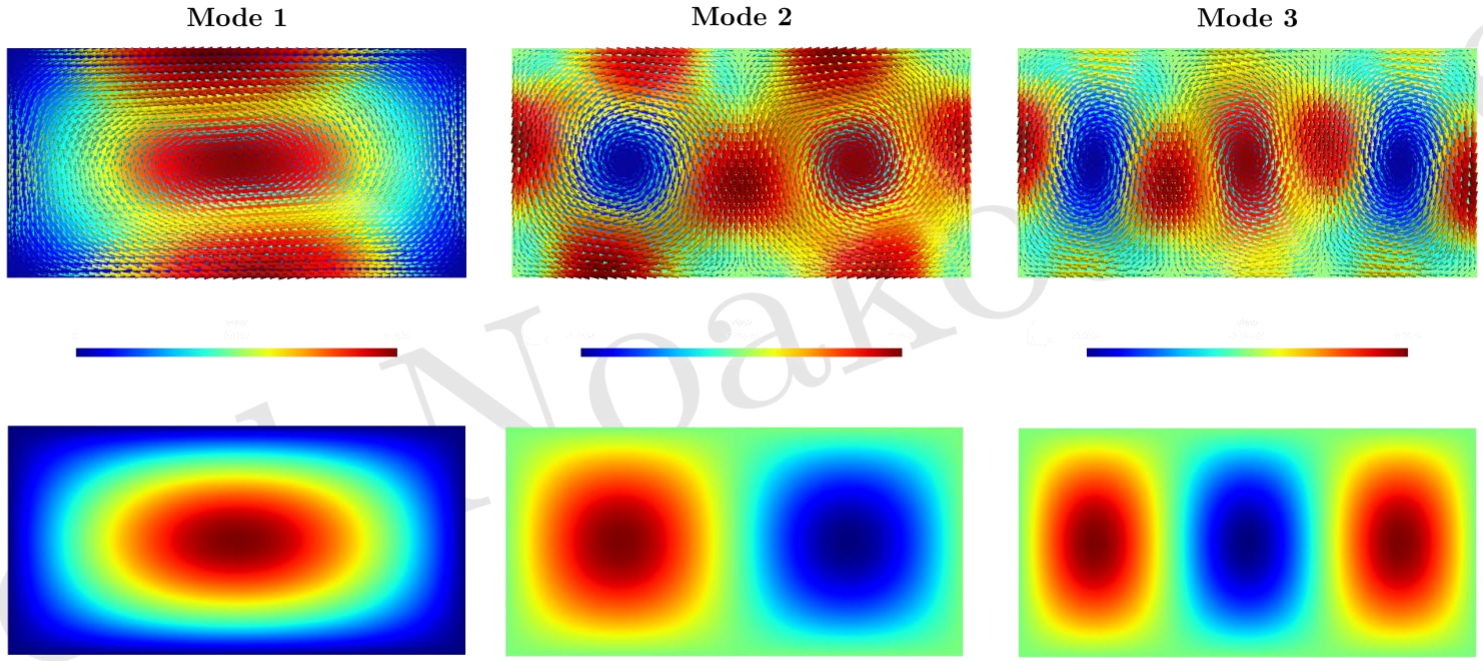
Computational EM
Finite Element Method (FEM)

TM^z Mode of 2D Rectangular Cavity



Analytic Solution

$$E_x = 0 \qquad H_x = \frac{1}{\mu}\frac{\partial A_z}{\partial y}$$

$$E_y = 0 \qquad H_y = -\frac{1}{\mu}\frac{\partial A_z}{\partial x}$$

$$E_z = -j\omega A_z \quad H_z = 0$$

$$A_z = C\sin\left(m\frac{x}{a}\pi\right)\sin\left(n\frac{y}{b}\pi\right)$$

$$f_r^{mn} = \frac{c_0}{2a\sqrt{\mu_r\epsilon_r}}\sqrt{m^2 + \left(n\frac{a}{b}\right)^2}$$

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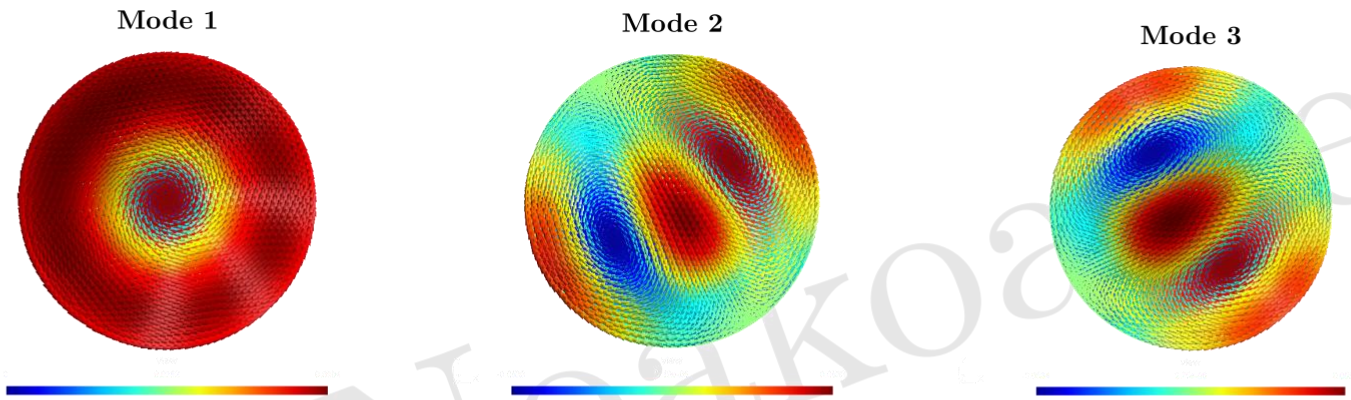
Computational EM
Finite Element Method (FEM)

TM^z Mode of 2D Circular Cavity



Analytic Solution

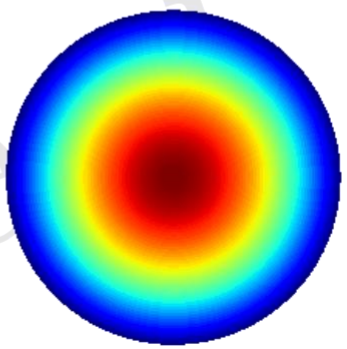
$$E_\rho = 0 \qquad H_\rho = \frac{1}{\mu}\frac{1}{\rho}\frac{\partial A_z}{\partial \phi}$$

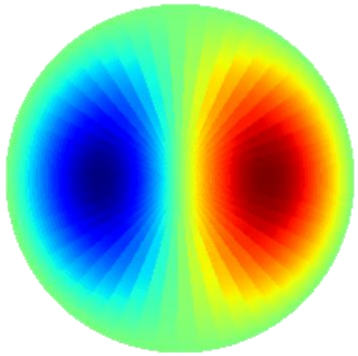
$$E_\phi = 0 \qquad H_\phi = -\frac{1}{\mu}\frac{\partial A_z}{\partial \rho}$$

$$E_z = -j\omega A_z \quad H_z = 0$$

$$A_z = CJ_m\left(\frac{\chi_{mn}}{\rho_0}\rho\right)\cos(m\phi)$$

$$f_r^{mn} = \frac{c_0}{2\pi\rho_0\sqrt{\mu_r\epsilon_r}}\chi_{mn}$$





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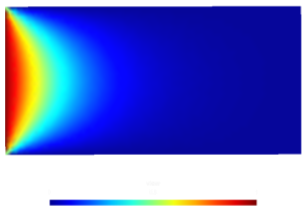
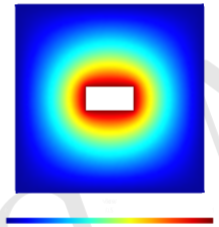
Computational EM
Finite Element Method (FEM)

2D Distribution of Electrostatic Potential

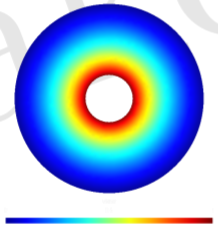
Potential Box



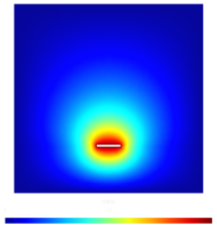
Stripline



Coax



Microstrip



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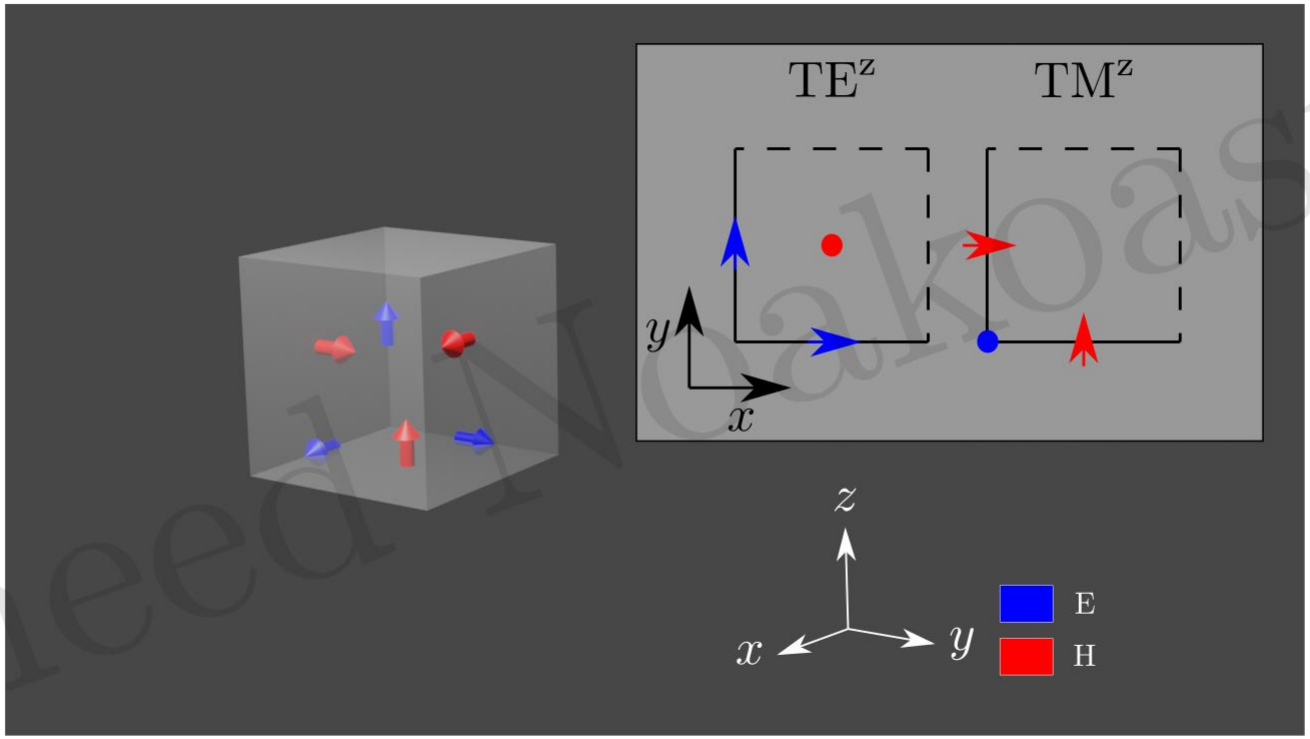
Computational EM
Finite Difference Time Domain (FDTD)

Yee Cell



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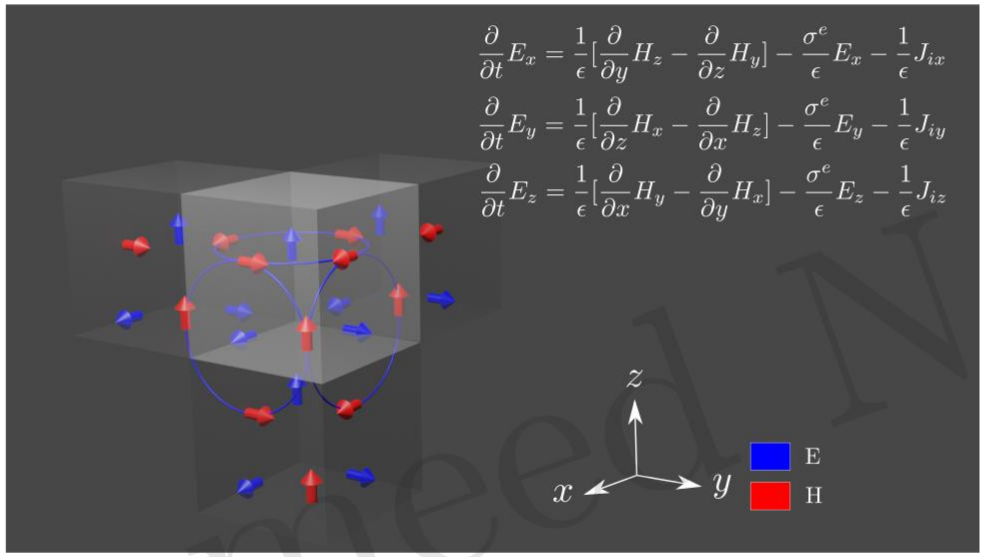
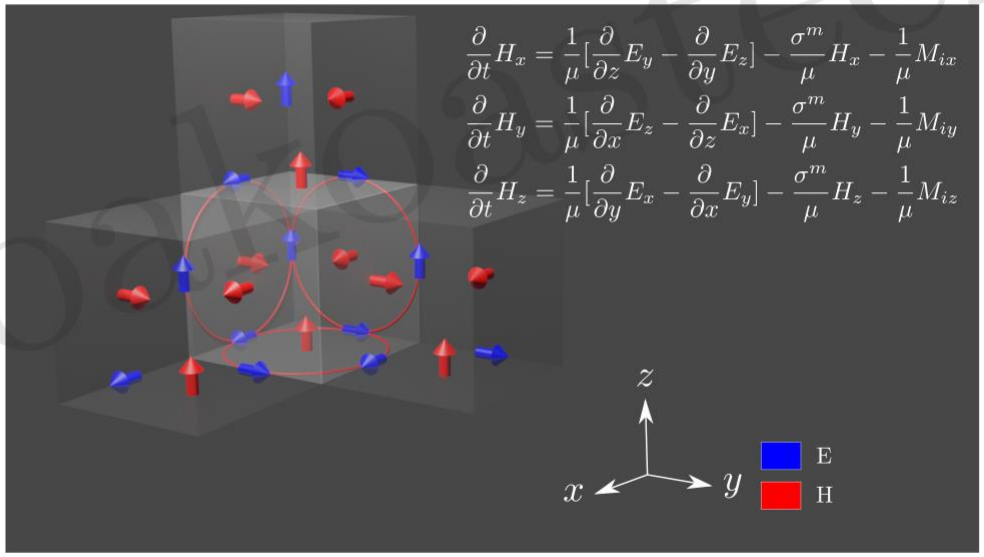
Computational EM
Finite Difference Time Domain (FDTD)

Yee Grid


$\frac{\partial}{\partial t}E_x = \frac{1}{\epsilon}[\frac{\partial}{\partial y}H_z - \frac{\partial}{\partial z}H_y] - \frac{\sigma^e}{\epsilon}E_x - \frac{1}{\epsilon}J_{ix}$
$\frac{\partial}{\partial t}E_y = \frac{1}{\epsilon}[\frac{\partial}{\partial z}H_x - \frac{\partial}{\partial x}H_z] - \frac{\sigma^e}{\epsilon}E_y - \frac{1}{\epsilon}J_{iy}$
$\frac{\partial}{\partial t}E_z = \frac{1}{\epsilon}[\frac{\partial}{\partial x}H_y - \frac{\partial}{\partial y}H_x] - \frac{\sigma^e}{\epsilon}E_z - \frac{1}{\epsilon}J_{iz}$
z
x
y
E
H


$\frac{\partial}{\partial t}H_x = \frac{1}{\mu}[\frac{\partial}{\partial z}E_y - \frac{\partial}{\partial y}E_z] - \frac{\sigma^m}{\mu}H_x - \frac{1}{\mu}M_{ix}$
$\frac{\partial}{\partial t}H_y = \frac{1}{\mu}[\frac{\partial}{\partial x}E_z - \frac{\partial}{\partial z}E_x] - \frac{\sigma^m}{\mu}H_y - \frac{1}{\mu}M_{iy}$
$\frac{\partial}{\partial t}H_z = \frac{1}{\mu}[\frac{\partial}{\partial y}E_x - \frac{\partial}{\partial x}E_y] - \frac{\sigma^m}{\mu}H_z - \frac{1}{\mu}M_{iz}$
z
x
y
E
H

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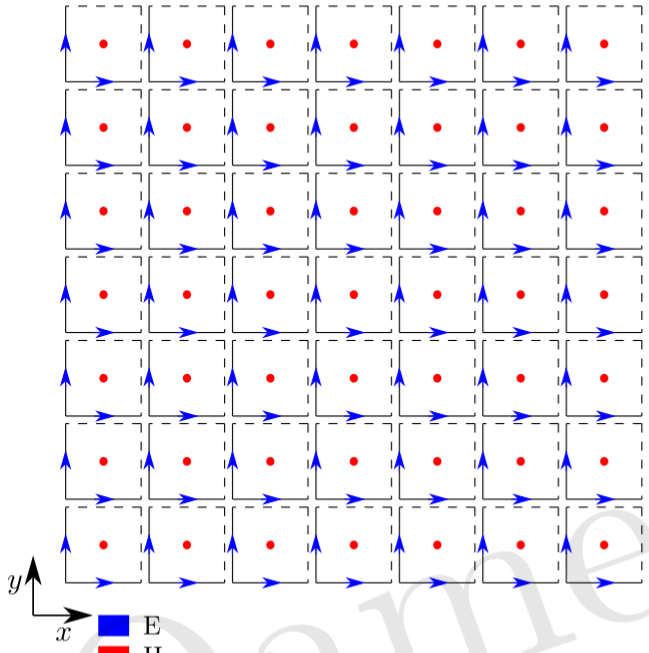
Computational EM
Finite Difference Time Domain (FDTD)

Update Equations for 2D TE^z Yee Grid



$$H_z^{n+\frac{1}{2}}(i,j) = C_{hh}(i,j)H_z^{n-\frac{1}{2}}(i,j) + C_{hex}(i,j)\left[E_x^n(i,j+1) - E_x^n(i,j)\right] + C_{hey}(i,j)\left[E_y^n(i+1,j) - E_y^n(i,j)\right] + C_{hm}(i,j)M_z^n(i,j)$$

$$E_x^{n+1}(i,j) = C_{exe}(i,j)E_x^n(i,j) + C_{exh}(i,j)\left[H_z^{n+\frac{1}{2}}(i,j) - H_z^{n+\frac{1}{2}}(i,j-1)\right] + C_{exj}(i,j)J_x^{n+\frac{1}{2}}(i,j)$$

$$E_y^{n+1}(i,j) = C_{eye}(i,j)E_y^n(i,j) + C_{eyh}(i,j)\left[H_z^{n+\frac{1}{2}}(i,j) - H_z^{n+\frac{1}{2}}(i-1,j)\right] + C_{eyj}(i,j)J_y^{n+\frac{1}{2}}(i,j)$$

$$C_{hh}(i,j) = \frac{2\mu(i,j) - \Delta t\sigma^m(i,j)}{2\mu(i,j) + \Delta t\sigma^m(i,j)}$$

$$C_{hex}(i,j) = \frac{2\Delta t}{[2\mu(i,j) + \Delta t\sigma^m(i,j)]\,\Delta y}$$

$$C_{hey}(i,j) = -\frac{2\Delta t}{[2\mu(i,j) + \Delta t\sigma^m(i,j)]\,\Delta x}$$

$$C_{hm}(i,j) = -\frac{2\Delta t}{2\mu(i,j) + \Delta t\sigma^m(i,j)}$$

$$C_{exe}(i,j) = \frac{2\epsilon(i,j) - \Delta t\sigma^e(i,j)}{2\epsilon(i,j) + \Delta t\sigma^e(i,j)}$$

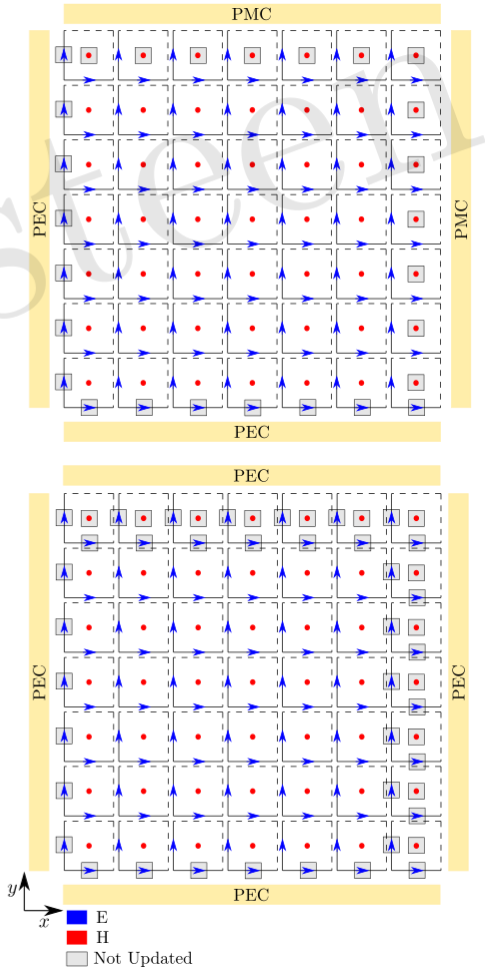
$$C_{exh}(i,j) = \frac{2\Delta t}{[2\epsilon(i,j) + \Delta t\sigma^e(i,j)]\,\Delta y}$$

$$C_{exj}(i,j) = -\frac{2\Delta t}{2\epsilon(i,j) + \Delta t\sigma^e(i,j)}$$

$$C_{eye}(i,j) = \frac{2\epsilon(i,j) - \Delta t\sigma^e(i,j)}{2\epsilon(i,j) + \Delta t\sigma^e(i,j)}$$

$$C_{eyh}(i,j) = -\frac{2\Delta t}{[2\epsilon(i,j) + \Delta t\sigma^e(i,j)]\,\Delta x}$$

$$C_{eyj}(i,j) = -\frac{2\Delta t}{2\epsilon(i,j) + \Delta t\sigma^e(i,j)}$$



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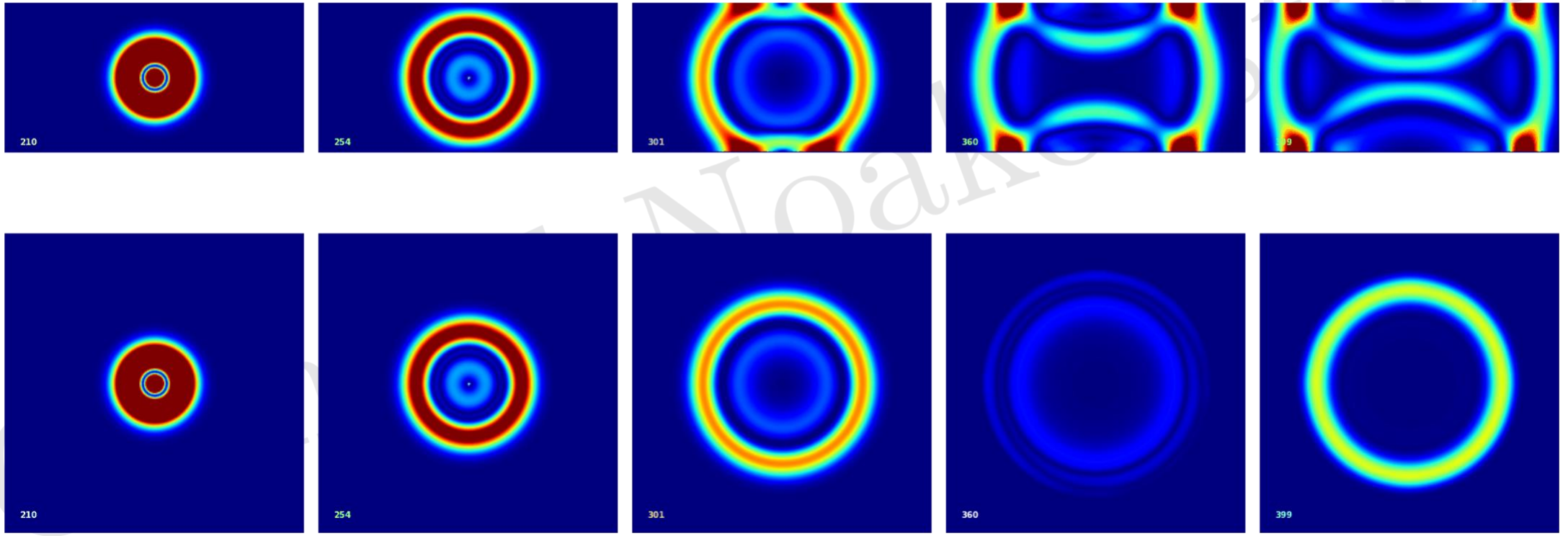
Computational EM
Finite Difference Time Domain (FDTD)

2D TE^z: Rectangular and Circular Cavities



Computational EM
Finite Difference Time Domain (FDTD)

Media with Anisotropy in Conductivities: 2D TE^z Mode

Maxwell's Equations

$$E_x = -j\frac{1}{\omega\epsilon S_y^e}\frac{\partial H_z}{\partial y} \qquad S_y^e \equiv 1 - j\frac{\sigma_y^e}{\omega\epsilon}$$

$$E_y = \quad j\frac{1}{\omega\epsilon S_x^e}\frac{\partial H_z}{\partial x} \qquad S_x^e \equiv 1 - j\frac{\sigma_x^e}{\omega\epsilon}$$

$$H_{zx} = \quad j\frac{1}{\omega\mu S_x^m}\frac{\partial E_y}{\partial x} \qquad S_x^m \equiv 1 - j\frac{\sigma_x^m}{\omega\mu}$$

$$H_{zy} = -j\frac{1}{\omega\mu S_y^m}\frac{\partial E_x}{\partial y} \qquad S_y^m \equiv 1 - j\frac{\sigma_y^m}{\omega\mu}$$

$$H_z = H_{zx} + H_{zy}$$

Solutions

$$\boldsymbol{E} = \frac{E_0}{W}\left(-\sqrt{\frac{S_y^m}{S_y^e}}\sin\theta\,\boldsymbol{a}_x + \sqrt{\frac{S_x^m}{S_x^e}}\cos\theta\,\boldsymbol{a}_y\right)e^{-j\boldsymbol{k}\cdot\boldsymbol{r}}$$

$$\boldsymbol{H} = \left(\frac{1}{Z}E_0\boldsymbol{a}_z\right)e^{-j\boldsymbol{k}\cdot\boldsymbol{r}}$$

$$\boldsymbol{k} = \omega\sqrt{\mu\epsilon}\left(\sqrt{S_x^e S_x^m}\cos\theta\,\boldsymbol{a}_x + \sqrt{S_y^e S_y^m}\sin\theta\,\boldsymbol{a}_y\right)$$

$$Z = \sqrt{\frac{\mu}{\epsilon}}W$$

$$W \equiv \sqrt{\frac{S_x^m}{S_x^e}\cos^2\theta + \frac{S_y^m}{S_y^e}\sin^2\theta}$$

Matched Medium

Matching Condition

$$\sigma_x^m = \frac{\mu}{\epsilon}\sigma_x^e$$

$$\sigma_y^m = \frac{\mu}{\epsilon}\sigma_y^e$$

Uniform Plane Wave

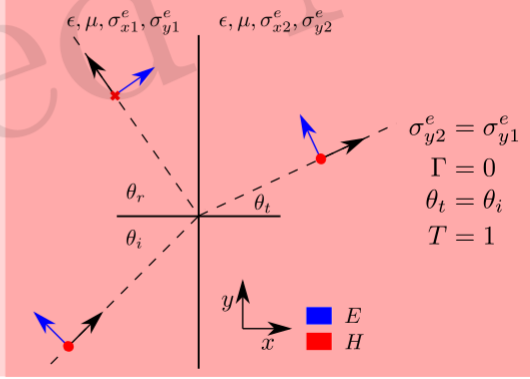
$$\boldsymbol{E} = \boldsymbol{E}_0 e^{-\sqrt{\frac{\mu}{\epsilon}}\left[\sigma_x^e\cos(\theta)x + \sigma_y^e\sin(\theta)y\right]}e^{-j\omega\sqrt{\mu\epsilon}\boldsymbol{a}_p\cdot\boldsymbol{r}}$$

$$\boldsymbol{H} = \frac{1}{\eta}\boldsymbol{a}_p \times \boldsymbol{E}$$

$$\boldsymbol{E}_0 \equiv E_0\left(-\sin\theta\,\boldsymbol{a}_x + \cos\theta\,\boldsymbol{a}_y\right)$$

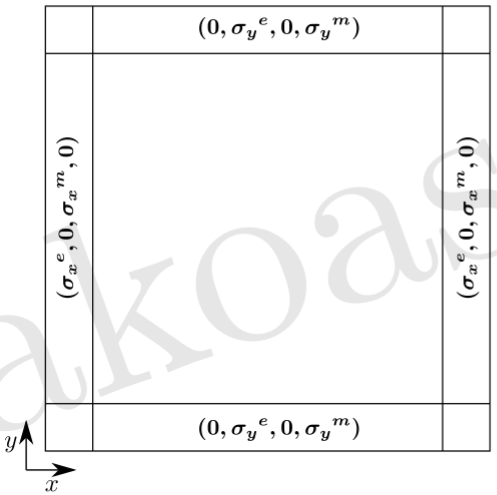
$$\boldsymbol{a}_p \equiv \cos\theta\,\boldsymbol{a}_x + \sin\theta\,\boldsymbol{a}_y$$

Reflection & Transmission



$$\sigma_{y2}^e = \sigma_{y1}^e$$
$$\Gamma = 0$$
$$\theta_t = \theta_i$$
$$T = 1$$

Split-PML ABC: 2D TE^z Mode



Electric Conductivity Profile

Left/Bottom

$$\sigma(n) = \sigma_{max}\left(\frac{n_e - n}{n_e - n_b}\right)^P$$

Right/Top

$$\sigma(n) = \sigma_{max}\left(\frac{n - n_b}{n_e - n_b}\right)^P$$

$$\sigma_{max} = -\sqrt{\frac{\epsilon}{\mu}}\frac{(P+1)\ln(R_0)}{2N\Delta}$$

n_b is the beginning index of the PML region
n_e is the ending index of the PML region
R_0 is the Required Attenuation
N is the number of cells in the PML region
Δ is the cell size

Modified H_z Update

$$H_{zx}^{n+\frac{1}{2}}(i,j) = C_{hh}(i,j)H_{zx}^{n-\frac{1}{2}}(i,j) + C_{hey}(i,j)\left[E_y^n(i+1,j) - E_y^n(i,j)\right]$$

$$C_{hh}(i,j) = \frac{2\mu(i,j) - \Delta t\sigma_x^m(i,j)}{2\mu(i,j) + \Delta t\sigma_x^m(i,j)}$$

$$C_{hey}(i,j) = -\frac{2\Delta t}{\left[2\mu(i,j) + \Delta t\sigma_x^m(i,j)\right]\Delta x}$$

$$H_{zy}^{n+\frac{1}{2}}(i,j) = C_{hh}(i,j)H_{zy}^{n-\frac{1}{2}}(i,j) + C_{hex}(i,j)\left[E_x^n(i,j+1) - E_x^n(i,j)\right]$$

$$C_{hh}(i,j) = \frac{2\mu(i,j) - \Delta t\sigma_y^m(i,j)}{2\mu(i,j) + \Delta t\sigma_y^m(i,j)}$$

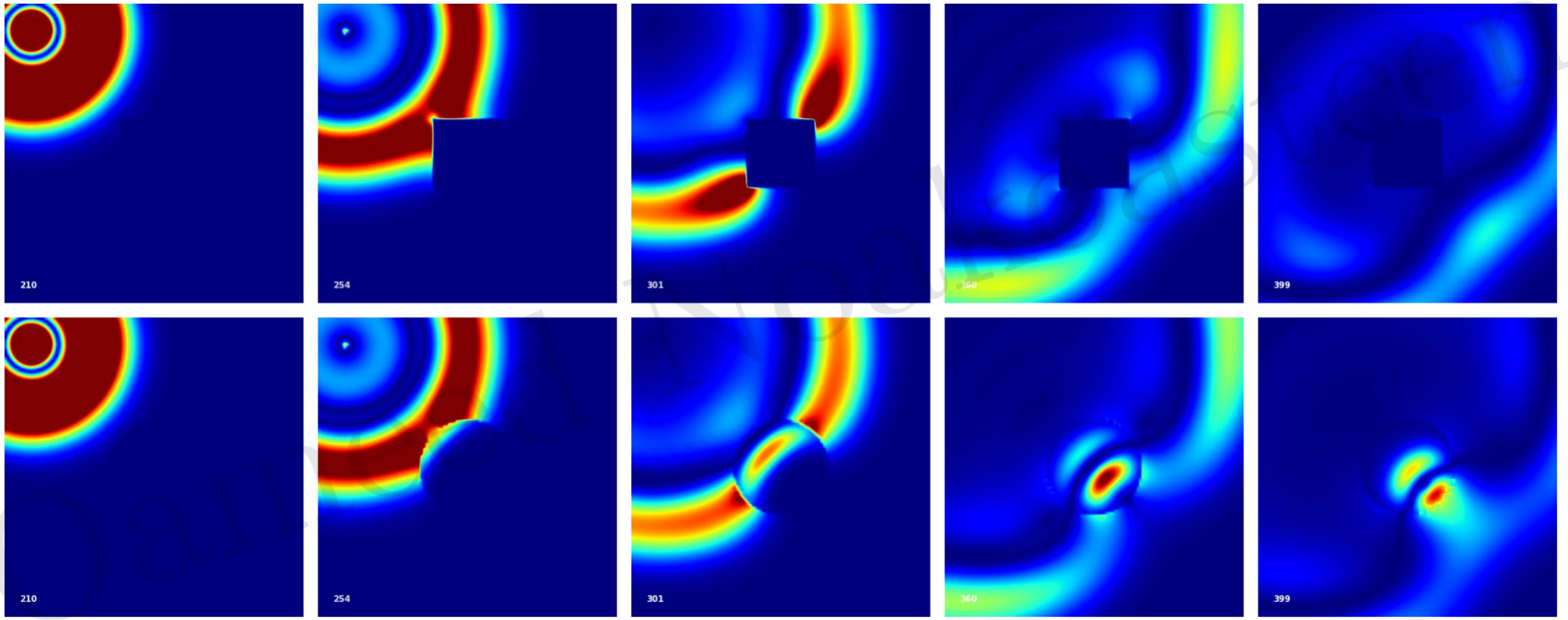
$$C_{hex}(i,j) = \frac{2\Delta t}{\left[2\mu(i,j) + \Delta t\sigma_y^m(i,j)\right]\Delta y}$$

$$H_z^{n+\frac{1}{2}}(i,j) = H_{zx}^{n+\frac{1}{2}}(i,j) + H_{zy}^{n+\frac{1}{2}}(i,j)$$

Cameed Noakoasteen

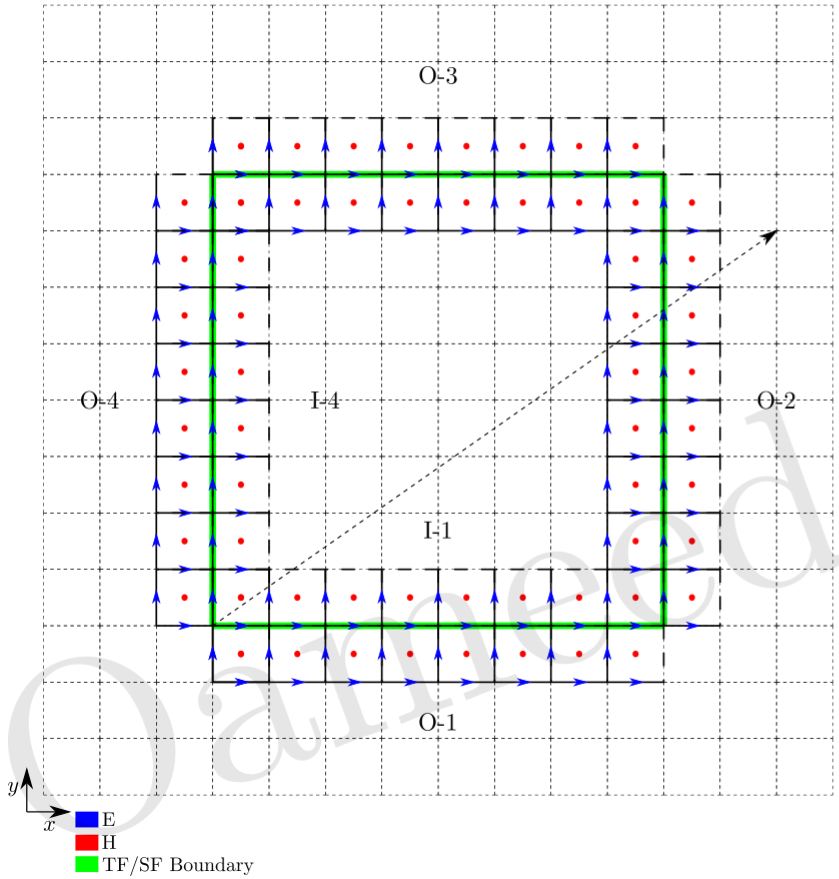
Computational EM
Finite Difference Time Domain (FDTD)

2D TE^z: Scattering form PEC and Dielectric Objects



Computational EM
Finite Difference Time Domain (FDTD)

TF/SF Excitation for 2D TE^z Mode



Consistency Updates

O-1 Region

$$H_z^{n+\frac{1}{2}}(i,j) = H_z^{n+\frac{1}{2}}(i,j) - C_{hex}(i,j)E^n_{x,\,inc}(i,j+1)$$

O-2 Region

$$E_y^{n+1}(i,j) = E_y^{n+1}(i,j) + C_{eyh}(i,j)H^{n+\frac{1}{2}}_{z,\,inc}(i,j)$$

$$H_z^{n+\frac{1}{2}}(i,j) = H_z^{n+\frac{1}{2}}(i,j) + C_{hey}(i,j)E^n_{y,\,inc}(i,j)$$

O-3 Region

$$E_x^{n+1}(i,j) = E_x^{n+1}(i,j) + C_{exh}(i,j)H^{n+\frac{1}{2}}_{z,\,inc}(i,j)$$

$$H_z^{n+\frac{1}{2}}(i,j) = H_z^{n+\frac{1}{2}}(i,j) + C_{hex}(i,j)E^n_{x,\,inc}(i,j)$$

O-4 Region

$$H_z^{n+\frac{1}{2}}(i,j) = H_z^{n+\frac{1}{2}}(i,j) - C_{hey}(i,j)E^n_{y,\,inc}(i+1,j)$$

I-1 Region

$$E_x^{n+1}(i,j) = E_x^{n+1}(i,j) - C_{exh}(i,j)H^{n+\frac{1}{2}}_{z,\,inc}(i,j-1)$$

I-4 Region

$$E_y^{n+1}(i,j) = E_y^{n+1}(i,j) - C_{eyh}(i,j)H^{n+\frac{1}{2}}_{z,\,inc}(i-1,j)$$

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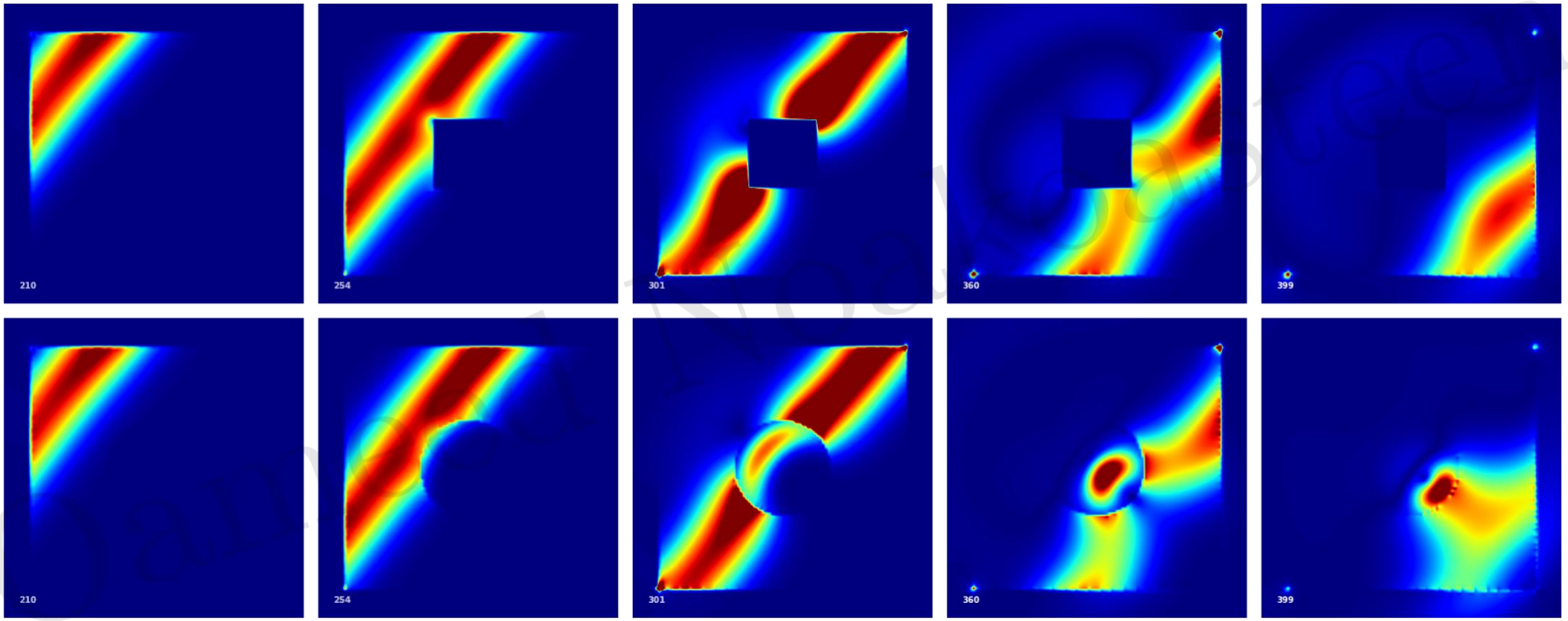
Computational EM
Finite Difference Time Domain (FDTD)

2D TE^z: Scattering form PEC and Dielectric Objects with TF/SF Excitation



Computational EM
Finite Difference Time Domain (FDTD)

Pulse Waveforms

Cosine Modulated Gaussian

$$g(t) = \cos\left(\omega_c(t - t_0)\right) e^{-\left(\frac{t-t_0}{\tau}\right)^2}$$

$$f_c,\ \text{BW} \equiv \frac{\Delta f}{f_c}$$

$$\tau = \frac{\sqrt{2.3}}{\pi\left(\frac{\Delta f}{2}\right)}$$

$$t_0 = \sqrt{20}\tau$$

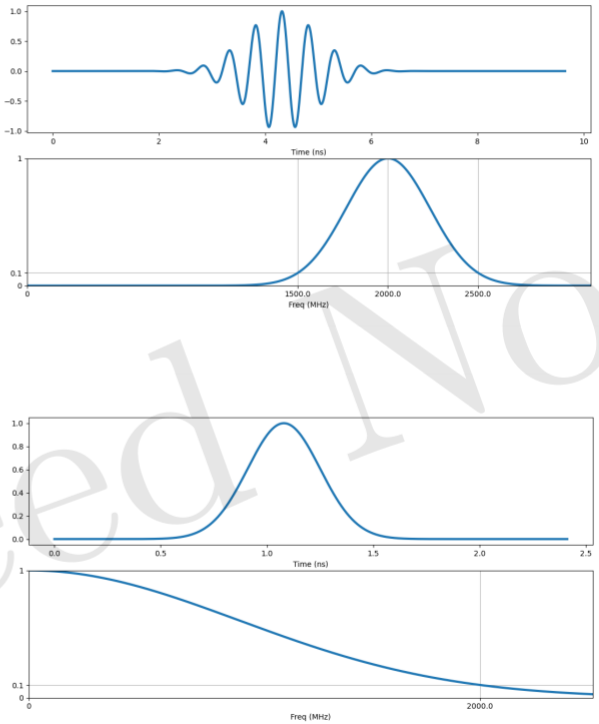
Gaussian

$$g(t) = e^{-\left(\frac{t-t_0}{\tau}\right)^2}$$

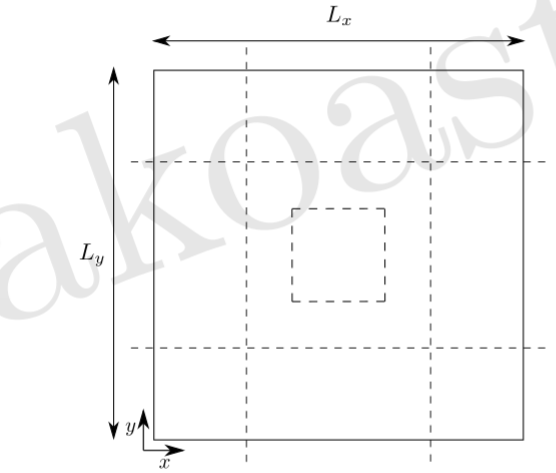
$$f_{max}$$

$$\tau = \frac{\sqrt{2.3}}{\pi f_{max}}$$

$$t_0 = \sqrt{20}\tau$$



Domain Design



- **Domain Size**

  For cavity problems, L_x and L_y are absolute values in meters.
  For propagation problems, $L_y = L_x = 10\lambda_{\min}$.

- **Number of Cells per Wavelength**

  $n_c \geq 20$

- **Spatial Discretization**

  $dx = \frac{\lambda_{\min}}{n_c}$, $dy = dx$

- **Courant–Friedrichs–Lewy (CFL) Number**

  $CFL \leq \frac{1}{\sqrt{D}}$ in which D is the dimensionality of the problem

- **Temporal Discretization**

  $dt = \frac{dx}{c} CFL$

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Computational EM
Finite Difference Time Domain (FDTD)

Update Equations for 1D TE^z Yee Grid

$$H_z^{n+\frac{1}{2}}(i) = C_{hh}(i)H_z^{n-\frac{1}{2}}(i) + C_{hey}(i)\left[E_y^n(i+1) - E_y^n(i)\right] + C_{hm}(i)M_z^n(i)$$

$$C_{hh}(i) = \frac{2\mu(i) - \Delta t\sigma^m(i)}{2\mu(i) + \Delta t\sigma^m(i)}$$

$$C_{hey}(i) = -\frac{2\Delta t}{[2\mu(i) + \Delta t\sigma^m(i)]\,\Delta x}$$

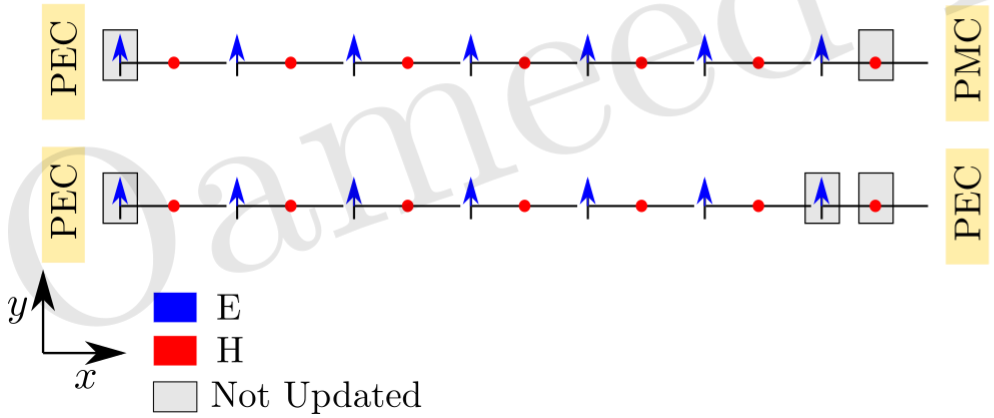
$$C_{hm}(i) = -\frac{2\Delta t}{2\mu(i) + \Delta t\sigma^m(i)}$$

$$E_y^{n+1}(i) = C_{eye}(i)E_y^n(i) + C_{eyh}(i)\left[H_z^{n+\frac{1}{2}}(i) - H_z^{n+\frac{1}{2}}(i-1)\right] + C_{eyj}(i)J_y^{n+\frac{1}{2}}(i)$$

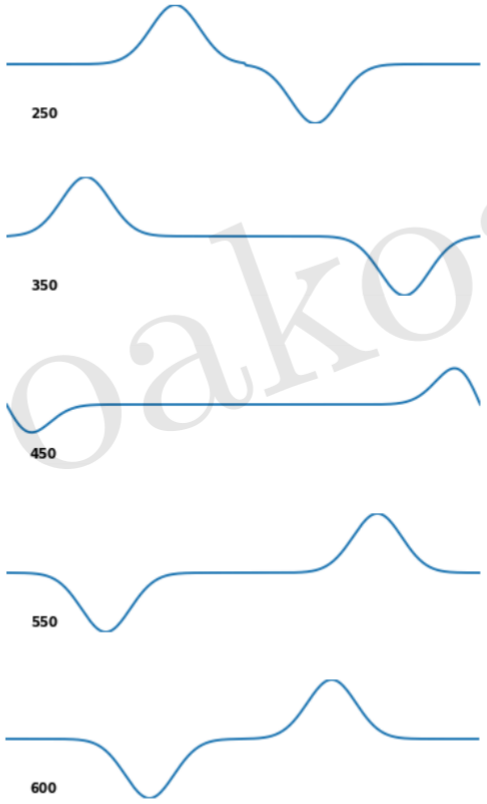
$$C_{eye}(i) = \frac{2\epsilon(i) - \Delta t\sigma^e(i)}{2\epsilon(i) + \Delta t\sigma^e(i)}$$

$$C_{eyh}(i) = -\frac{2\Delta t}{[2\epsilon(i) + \Delta t\sigma^e(i)]\,\Delta x}$$

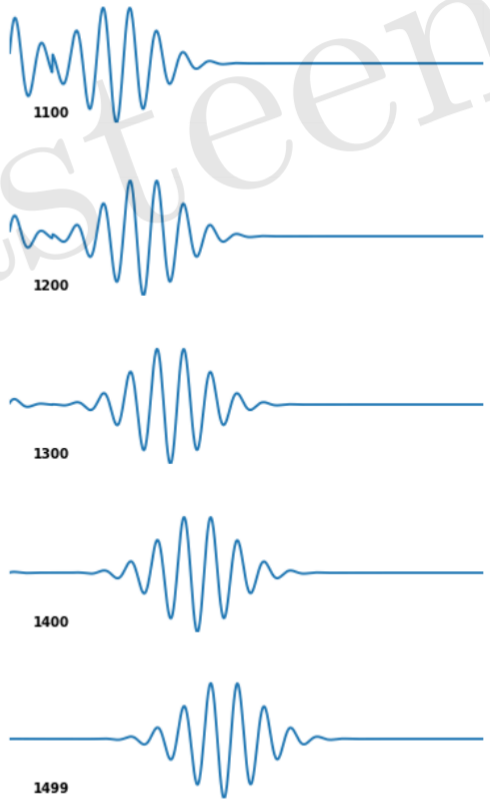
$$C_{eyj}(i) = -\frac{2\Delta t}{2\epsilon(i) + \Delta t\sigma^e(i)}$$



1D TE^z Cavity



1D TE^z Free Space



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Computational EM
Method of Moments (MoM)

Point Matching & Piecewise Constant Basis Functions

■ EFIE for TM^z Polarized Wave

$$\frac{\eta k}{4}\int_C J_s\left(\boldsymbol{\rho}_b^{'}\right) H_0^{(2)}\left(k\left|\boldsymbol{\rho}_b-\boldsymbol{\rho}_b^{'}\right|\right) dC^{'} = E_{inc}\left(\boldsymbol{\rho}=\boldsymbol{\rho}_b\right)$$

Coefficients Matrix $\boldsymbol{Z}$

Self Terms:

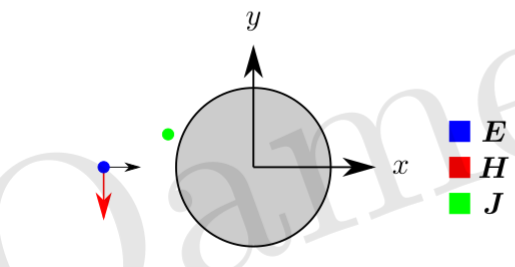
$$Z_{i,i} = \frac{\eta k}{4}\left[1+j\frac{2}{\pi}\left(1+\frac{1}{288}\left(\Delta\phi\right)^2-\ln\left(\frac{1.781k\rho_b}{4}\Delta\phi\right)\right)\right]\Delta\phi$$

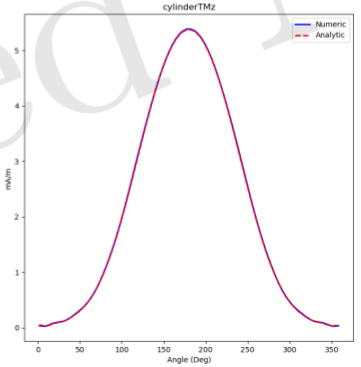
Non-Self Terms:

$$Z_{i,j} = \frac{\eta k}{4}\int_{\Omega_e^j} H_0^{(2)}\left(k\left|\boldsymbol{\rho}_{b,i}-\boldsymbol{\rho}^{'}\right|\right) dC^{'}$$

Forcing Function Vector $\boldsymbol{f}$

$$f_i = E_0 e^{-jk\boldsymbol{a}_{inc}\cdot\boldsymbol{\rho}_{b,i}}$$





Analytic Solution

$$J_z\left(\phi\right) = \frac{2E_0}{\pi k\eta\rho_b}\sum_{n=-\infty}^{+\infty}\left(j\right)^{-n}\frac{1}{H_n^{(2)}\left(k\rho_b\right)}e^{jn\phi}$$

■ MFIE for TE^z Polarized Wave

$$J_s(\boldsymbol{\rho}_b) - j\frac{1}{4}\int_C J_s\left(\boldsymbol{\rho}_b^{'}\right)\left(\boldsymbol{\nabla}^{'}H_0^{(2)}\left(k\left|\boldsymbol{\rho}-\boldsymbol{\rho}^{'}\right|\right)\cdot\boldsymbol{n}^{'}\right) dC^{'} = -H_{inc}\left(\boldsymbol{\rho}=\boldsymbol{\rho}_b\right)$$

Coefficients Matrix $\boldsymbol{Z}$

Self Terms:

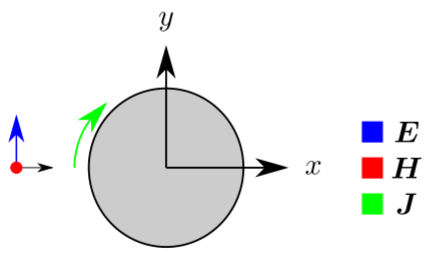
$$Z_{i,i} = \frac{1}{2}$$

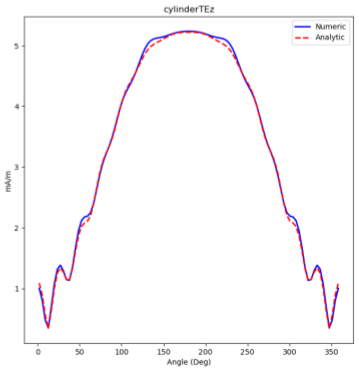
Non-Self Terms:

$$Z_{i,j} = j\frac{k}{4}\int_{\Omega_e^j} H_1^{(2)}\left(k\left|\boldsymbol{\rho}_{b,i}-\boldsymbol{\rho}^{'}\right|\right)\cos\xi^{'}\,dC^{'},\ \cos\xi^{'} = -\frac{1}{\sqrt{2}}\sqrt{1-\cos\left(\phi^{'}-\phi_i\right)}$$

Forcing Function Vector $\boldsymbol{f}$

$$f_i = -\frac{1}{\eta}\left(E_{0y}\cos\phi_{inc} - E_{0x}\sin\phi_{inc}\right)e^{-jk\boldsymbol{a}_{inc}\cdot\boldsymbol{\rho}_{b,i}}$$





Analytic Solution

$$J_z\left(\phi\right) = j\frac{2H_0}{\pi k\rho_b}\sum_{n=-\infty}^{+\infty}\left(j\right)^{-n}\frac{1}{H_n^{(2)^{'}}\left(k\rho_b\right)}e^{jn\phi}$$

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Computational EM
Method of Moments (MoM)

Point Matching & Piecewise Constant Basis Functions

■ Conducting Plate Held at Constant Potential

$$\frac{1}{4\pi\epsilon}\iint \rho\left(x',y'\right)\frac{1}{\sqrt{(x-x')^2+(y-y')^2}}\,dx'dy' = V_0$$

Coefficients Matrix Z

Self Terms:

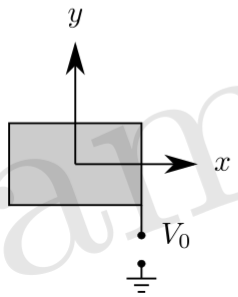
$$Z_{i,i} = \frac{\Delta}{2\pi\epsilon}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

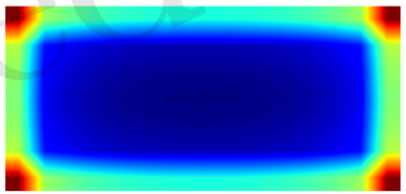
Non-Self Terms:

$$Z_{i,j} = \frac{1}{4\pi\epsilon}\iint_{\Omega_e^j}\frac{1}{\sqrt{(x_i-x')^2+(y_i-y')^2}}\,dx'dy'$$

Forcing Function Vector f

$$f_i = V_0$$





■ Conducting Rod Held at Constant Potential

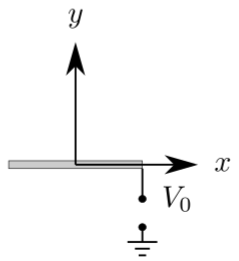
$$\frac{1}{4\pi\epsilon}\int \rho\left(x'\right)\frac{1}{\sqrt{(x-x')^2+h^2}}\,dx' = V_0$$

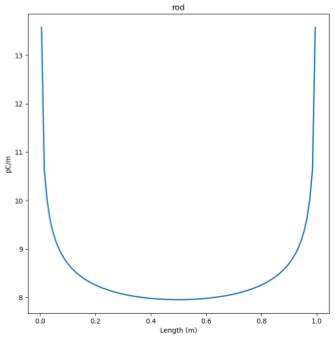
Coefficients Matrix Z

$$Z_{i,j} = \frac{1}{4\pi\epsilon}\int_{\Omega_e^j}\frac{1}{\sqrt{(x_i-x')^2+h^2}}\,dx'$$

Forcing Function Vector f

$$f_i = V_0$$





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Appendices
Radiation of a Line Source

Hypothetical Source Distribution

■ **Infinite Line of Electric Current**

$$\boldsymbol{J}(\boldsymbol{r}) = I_0 \delta(\rho - \rho_0)\left(\frac{1}{\rho}\delta(\phi - \phi_0)\right)\boldsymbol{a}_z$$

$$\boldsymbol{A} = A_z \boldsymbol{a}_z,\ A_z(\boldsymbol{\rho}) = -j\frac{\mu}{4} I_0 H_0^{(2)}(k|\boldsymbol{\rho} - \boldsymbol{\rho}_0|)$$

Radiation (The Complete Solution)

■ **The TM^z Mode**

$$E_\rho = 0 \qquad H_\rho = j\frac{k}{4} I_0 \frac{H_1^{(2)}(k|\boldsymbol{\rho} - \boldsymbol{\rho}_0|)}{|\boldsymbol{\rho} - \boldsymbol{\rho}_0|}(\rho_0 \sin\Delta\phi)$$

$$E_\phi = 0 \qquad H_\phi = -j\frac{k}{4} I_0 \frac{H_1^{(2)}(k|\boldsymbol{\rho} - \boldsymbol{\rho}_0|)}{|\boldsymbol{\rho} - \boldsymbol{\rho}_0|}(\rho - \rho_0 \cos\Delta\phi)$$

$$E_z = -\frac{k\eta}{4} I_0 H_0^{(2)}(k|\boldsymbol{\rho} - \boldsymbol{\rho}_0|) \qquad H_z = 0$$

Radiation in the Far-Field Region

■ **The TEM$^\rho$ Mode**

$$E_\rho = 0$$

$$E_\phi = 0$$

$$E_z = -\left(\eta\sqrt{\frac{k}{8\pi}} e^{j\left(k\boldsymbol{a}_\rho \cdot \boldsymbol{\rho}_0 + \frac{\pi}{4}\right)}\right) I_0 \frac{1}{\sqrt{\rho}} e^{-jk\rho}$$

$$\boldsymbol{H} = \frac{1}{j\eta}\boldsymbol{a}_\rho \times \boldsymbol{E}$$

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Appendices
Propagation in Magnetic Media

Ferrite Media

- **Scenario I:**
  Propagation in the Direction of the Magnetic Bias
  - The Bias Magnetic Field

$$\boldsymbol{H}_0^{\text{bias}} = H_0^{\text{bias}} \boldsymbol{a}_z, \ \boldsymbol{\mu} = \begin{bmatrix} \mu & j\kappa & 0 \\ -j\kappa & \mu & 0 \\ 0 & 0 & \mu_0 \end{bmatrix}$$

  - The wave is propagating in the positive direction along the z axis.
  - The Uniform Plane Wave

$$\boldsymbol{E} = \boldsymbol{E}_0^{\text{RHCP}} e^{-j\beta^{\text{RHCP}} z} + \boldsymbol{E}_0^{\text{LHCP}} e^{-j\beta^{\text{LHCP}} z}$$

$$\boldsymbol{H} = \frac{1}{\eta^{\text{RHCP}}} \boldsymbol{a}_z \times \boldsymbol{E}^{\text{RHCP}} + \frac{1}{\eta^{\text{LHCP}}} \boldsymbol{a}_z \times \boldsymbol{E}^{\text{LHCP}}$$

$$\boldsymbol{E}_0^{\text{RHCP}} = E_0 \left(j\boldsymbol{a}_x + \boldsymbol{a}_y \right), \ \beta^{\text{RHCP}} \equiv \omega \sqrt{\mu_{\text{eff}}^{\text{RHCP}} \epsilon}, \ \eta^{\text{RHCP}} \equiv \sqrt{\frac{\mu_{\text{eff}}^{\text{RHCP}}}{\epsilon}}, \ \mu_{\text{eff}}^{\text{RHCP}} \equiv \mu + \kappa$$

$$\boldsymbol{E}_0^{\text{LHCP}} = -E_0 \left(j\boldsymbol{a}_x - \boldsymbol{a}_y \right), \ \beta^{\text{LHCP}} \equiv \omega \sqrt{\mu_{\text{eff}}^{\text{LHCP}} \epsilon}, \ \eta^{\text{LHCP}} \equiv \sqrt{\frac{\mu_{\text{eff}}^{\text{LHCP}}}{\epsilon}}, \ \mu_{\text{eff}}^{\text{LHCP}} \equiv \mu - \kappa$$

  - The Faraday Rotation

$$\boldsymbol{E} = E_0 \left[\sin\left(\frac{1}{2} \beta^{\Delta} z \right) \boldsymbol{a}_x + \cos\left(\frac{1}{2} \beta^{\Delta} z \right) \boldsymbol{a}_y \right] e^{-j\frac{1}{2}\beta^{\Sigma} z}, \ \beta^{\Delta} \equiv \beta^{\text{RHCP}} - \beta^{\text{LHCP}}, \ \beta^{\Sigma} \equiv \beta^{\text{RHCP}} + \beta^{\text{LHCP}}$$

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Appendices
Electrostatics and Magnetostatics

Maxwell's Equations

Differential Form

$$\nabla \cdot \boldsymbol{D} = \rho$$

$$\nabla \times \boldsymbol{E} = \boldsymbol{0}$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}$$

Integral Form

$$\oiint_S \boldsymbol{D} \cdot d\boldsymbol{S} = Q$$

$$\oint_C \boldsymbol{E} \cdot d\boldsymbol{l} = 0$$

$$\oiint_S \boldsymbol{B} \cdot d\boldsymbol{S} = 0$$

$$\oint_C \boldsymbol{H} \cdot d\boldsymbol{l} = \iint_S \boldsymbol{J} \cdot d\boldsymbol{S}$$

Boundary Conditions

$$\boldsymbol{n}_{12} \cdot (\boldsymbol{D}_2 - \boldsymbol{D}_1) = \rho_s$$
$$\boldsymbol{n}_{12} \cdot (\boldsymbol{B}_2 - \boldsymbol{B}_1) = 0$$
$$\boldsymbol{n}_{12} \times (\boldsymbol{E}_2 - \boldsymbol{E}_1) = \boldsymbol{0}$$
$$\boldsymbol{n}_{12} \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = \boldsymbol{J}_s$$

Constitutive Relations

$$\boldsymbol{D} = \epsilon \boldsymbol{E},\ \boldsymbol{B} = \mu \boldsymbol{H},\ \boldsymbol{J} = \sigma \boldsymbol{E}$$

Complete Solutions

$$\boldsymbol{E} = -\nabla V$$

$$\boldsymbol{H} = \frac{1}{\mu} \nabla \times \boldsymbol{A}$$

- The Potentials

$$\nabla^2 V = -\frac{1}{\epsilon} \rho$$

$$\nabla^2 \boldsymbol{A} = -\mu \boldsymbol{J}$$

The scalar potential V and the vector potential $\boldsymbol{A}$ are solutions to their respective **poisson equation**.

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Appendices
Electrostatics and Magnetostatics

The Field Generated by a Source

■ $\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon}\iiint_{V'}\rho\left(\boldsymbol{r}'\right)\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\,dv'$

$\boldsymbol{H}(\boldsymbol{r}) = \frac{1}{4\pi}\iiint_{V'}\boldsymbol{J}\left(\boldsymbol{r}'\right)\times\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\,dv'$

Hypothetical Source Distributions

■ **Impulse of Charge Density (Point Charge)**

$\rho(\boldsymbol{r}) = q\delta(x)\,\delta(y)\,\delta(z)$

$\boldsymbol{E} = \frac{1}{4\pi\epsilon}q\frac{1}{r^2}\boldsymbol{a}_r$

■ **Infinite Line of Charge Density (Line Charge)**

$\rho(\boldsymbol{r}) = \rho_0\delta(\rho)\left(\frac{1}{\rho}\delta(\phi)\right)$

$\boldsymbol{E} = \frac{1}{2\pi\epsilon}\rho_0\frac{1}{\rho}\boldsymbol{a}_\rho$

■ **Electric Dipole**

$\rho(\boldsymbol{r}) = q\delta(x)\,\delta(y)\,\delta\left(z-\frac{d}{2}\right) - q\delta(x)\,\delta(y)\,\delta\left(z+\frac{d}{2}\right)$

$\boldsymbol{E} = \frac{1}{4\pi\epsilon}\frac{1}{r^3}\left[3\left(\boldsymbol{P}\cdot\boldsymbol{a}_r\right)\boldsymbol{a}_r - \boldsymbol{P}\right],\ \boldsymbol{P}\equiv qd\boldsymbol{a}_z,\ \frac{d}{r}\ll 1$

■ **Infinite Line of Current Density**

$\boldsymbol{J}(\boldsymbol{r}) = I_0\delta(\rho)\left(\frac{1}{\rho}\delta(\phi)\right)\boldsymbol{a}_z$

$\boldsymbol{H} = \frac{1}{2\pi}I_0\frac{1}{\rho}\boldsymbol{a}_\phi$

■ **Magnetic Dipole**

$\boldsymbol{J}(\boldsymbol{r}) = I_0\delta(\rho-\rho_0)\,\delta(z)\,\boldsymbol{a}_\phi$

$\boldsymbol{H} = \frac{1}{4\pi}\frac{1}{r^3}\left[3\left(\boldsymbol{M}\cdot\boldsymbol{a}_r\right)\boldsymbol{a}_r - \boldsymbol{M}\right],\ \boldsymbol{M}\equiv I_0\left(\pi\rho_0^2\right)\boldsymbol{a}_z,\ \frac{\rho_0}{r}\ll 1$

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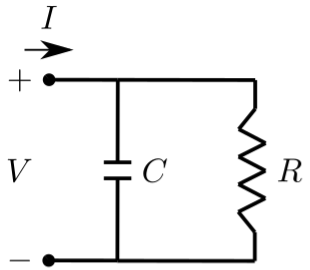
Appendices
RC, RL and RLC Filters

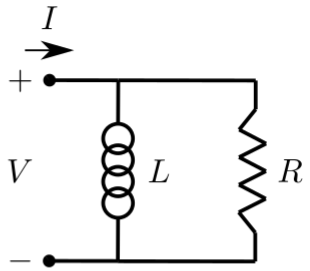
Lowpass



$$Z(S) = \frac{1}{C}\frac{1}{S+\frac{1}{RC}}$$

$$K = \frac{1}{C},\ \tau = RC$$

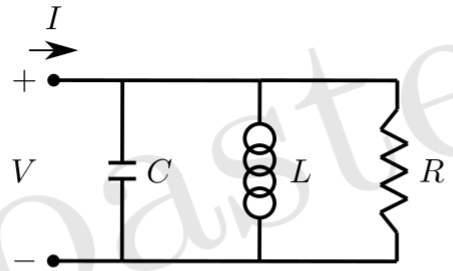
Highpass



$$Z(S) = R\frac{S}{S+\frac{R}{L}}$$

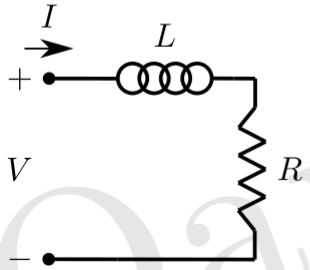
$$K = R,\ \tau = \frac{L}{R}$$

Bandpass



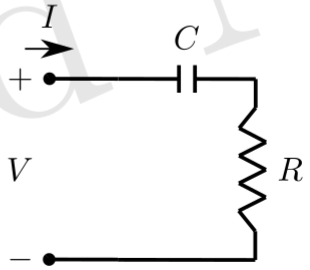
$$Z(S) = \frac{1}{C}\frac{S}{S^2+\frac{1}{RC}S+\frac{1}{LC}}$$

$$K = \frac{1}{C},\ \omega_0 = \frac{1}{\sqrt{LC}},\ Q = R\sqrt{\frac{C}{L}}$$



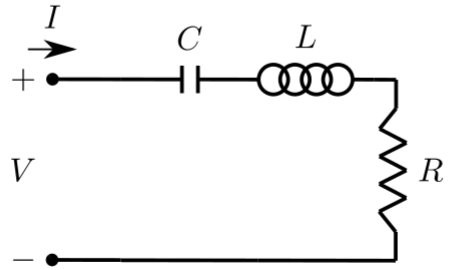
$$Y(S) = \frac{1}{L}\frac{1}{S+\frac{R}{L}}$$

$$K = \frac{1}{L},\ \tau = \frac{L}{R}$$



$$Y(S) = \frac{1}{R}\frac{S}{S+\frac{1}{RC}}$$

$$K = \frac{1}{R},\ \tau = \frac{R}{C}$$



$$Y(S) = \frac{1}{L}\frac{S}{S^2+\frac{R}{L}S+\frac{1}{LC}}$$

$$K = \frac{1}{L},\ \omega_0 = \frac{1}{\sqrt{LC}},\ Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$

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